Far East J.Theo.Stat. 31(2) (2010), 107-116

DEFINITE INTEGRALS, SOME INVOLVING RESIDUE THEORY EVALUATED BY THE MAPLE CODE

K.O. BOWMAN[1] and L.R. SHENTON[2]

[1]Computational Sciences and Engineering Division
Oak Ridge National Laboratory, P.O.Box 2008, 4500N, MS-6191
Oak Ridge, TN 37831-6191, U.S.A.
e-mail: bowmanko@ornl.gov
[2]Department of Statistics, University of Georgia
Athens, Georgia 30602, U.S.A.

March 3, 2010

Abstract

The calculus of residue is applied to evaluate certain integrals in the range (-∞ to ∞) using the Maple symbolic code. These integrals are of the form $\int_{-\infty}^{\infty} \cos(x)/[(x^2 + a^2)(x^2 + b^2)(x^2 + c^2)]dx$ and similar extensions. The Maple code is also applied to expressions in maximum likelihood estimator moments when sampling from the negative binomial distribution. In general the Maple code approach to the integrals gives correct answers to specified decimal places, but the symbolic result may be extremely long and complex.

2000 Mathematics Subject Classification: 62E20.
Key words and phrases; alternant determinant, calculus residues, contour integration.

*Notice: This manuscript has been authored by UT-Battelle, LLC, under contract DE-AC-05-00OR22725 with the U.S. Department of Energy.

1 Introduction

Interest in the theory of complex integrations by reading Copson (1935), Shenton wrote a note to Copson (in 1957) indicating that the statement of the fundamental residue theorem seemed incomplete, at least to him. Out of interest, Copson's letter which came from Harvard University is given in an Appendix 1, Copson describes the residue formed as being "very difficult". The reader should note that Copson basically agree with Shenton's points.

Note, we have already used the Maple code in connection with the skewness for maximum likelihood estimators of the negative binomial distribution.

Here we will consider definite integrals (range $-\infty$ to ∞) mentioned by Copson (1935, Chapter VI, The Calculus of Residues, pp.115-179, in perturbation pages 151 onwards). They will be generalized somewhat, to emphasize the power of the Maple code. However, we also refer to an unusual example arising in statistical theory of the negative binomial distribution.

2 Some Examples

2.1 Copson's cases

Example 1 (Copson (1935, p129)

$$M_1(a) = \int_{-\infty}^{\infty} \frac{\cos(x)dx}{x^2 + a^2} \qquad (a > 0)$$

and from Maple

$$M_1(a) = \pi e^{-a}/a$$

As a check,

$$M_1(a = 2) = 0.2125841654.$$

Example 2 (Copson (1935, p152)

$$M_2(a) = \int_{-\infty}^{\infty} \frac{\cos(x)dx}{(x^2 + a^2)(x^2 + b^2)} \qquad (a, b > 0, a \neq b)$$

and by Maple

$$M_2(a) = \frac{\pi(b\sinh(a) - b\cosh(a) - a\sinh(b) + a\cosh(b))}{ab(a^2 - b^2)} = \frac{\pi(ae^{-b} - be^{-a})}{ab(a^2 - b^2)}$$

and a check,

$$M_2(a=2, b=1) = 0.3143810613.$$

There is a generalization of examples 1 and 2. We consider the case

$$M(\underline{a}, n) = \int_{-\infty}^{\infty} \frac{cos(x)dx}{(x^2+a_1^2)(x^2+a_2^2)\cdots(x^2+a_n^2)}. \tag{1}$$

$a_1, a_2, \cdots, a_n$ being real, positive, and distinct. The integral can be expressed in determinantal form. Since each factor in the denominator is of the first power. In the contour integral residues arise at $a_1 i_1, a_2 i_2, \cdots, a_n i_n$. We find $M(\underline{a}, n) = \pi \mathbf{M}(\underline{a}, n)/\Delta(n)$ where

$$\mathbf{M}(\underline{a}, n) = (-1)^{n-1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ a_1^4 & a_2^4 & \cdots & a_n^4 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{2n-4} & a_2^{2n-4} & \cdots & a_n^{2n-4} \\ \frac{e^{-a_1}}{a_1} & \frac{e^{-a_2}}{a_2} & \cdots & \frac{e^{-a_n}}{a_n} \end{vmatrix}$$

and

$$\Delta(n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ a_1^4 & a_2^4 & \cdots & a_n^4 \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{2n-4} & a_2^{2n-4} & \cdots & a_n^{2n-4} \\ a_1^{2n-2} & a_2^{2n-2} & \cdots & a_n^{2n-2} \end{vmatrix}$$

both determinants being of alternant form.

A similar case arises when the integral is $x \sin x$, the last row in the M$(\underline{a}, n)$ being replaced by $e^{-a_1} \quad e^{-a_2} \quad \cdots \quad e^{-a_n}$. There will be many more cases of this alternant type.

For a reference to alternants see Aiken (1946), Bowman and Shenton (2003, 2004). A classical study of the subject of determinants is due to Sir Tomas Muir (1930).

Numerical examples are shown in Table 1.

Table 1 Numerical examples of equation (1)

Numerator	$\cos(x)$		$x\sin(x)$	
Case	1	2	1	2
$n = 2$	0.31438	518.86	0.24352	9.0170
$n = 3$	0.03529	5210.02	0.02372	51.5210
$n = 4$	0.002204	31042.44	0.00136	247.2823
$n = 5$	0.0	0.0	0.00005	860.2906

The case 1 represent $a_1 = 1$, $a_2 = 2$, $a_3 = 3 \cdots$, and the case 2 represent $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.3 \cdots$.

Example 3 (Copson (1935), p152)

$$M_3(a) = \int_{-\infty}^{\infty} \frac{\cos(x)dx}{(x^2 + a^2)^2} \qquad (a > 0)$$

and by Maple

$$M_3(a) = \frac{\pi(-\sinh(a) + \cosh(a) + a\cosh(a) - a\sinh(a))}{2a^3} = \frac{\pi(a+1)}{2a^3 e^a}$$

and a check

$$M_3(a = 2) = 0.07971906203.$$

Example 4 (Copson (1935), p.130)

$$M_4(a) = \int_{-\infty}^{\infty} \frac{x\sin(x)dx}{(x^2 + a^2)} \qquad (a > 0)$$

and by Maple

$$M_4(a) = \pi(\cosh(a) - \sinh(a)) = \pi e^{-a}$$

and a check,

$$M_4(a = 2) = 0.425168340.$$

Examples 1 through 4 agree with Copson's answers.

2.2 Some new examples of integrals using Maple

Example 5

$$M_5(a) = \int_{-\infty}^{\infty} \frac{(\cos(x))^s dx}{(x^2 + a^2)} \qquad (a > 0, s = 2, 3, 4)$$

For $s = 2$ Maple gives

$$M_5(a) = \frac{\pi(-2\sinh(a)\cosh(a) + 4\cosh(a)^2 - \sinh(2a))}{4a}$$

and by Maple

$$M_5(a = 2) = 0.7997832335.$$

For $s = 3$ Maple gives

$$M_5(a) = -\frac{\pi(4\sinh(a)\cosh(a)^2 + 5\sinh(a) - 8\cosh(a)^3 + \sinh(3a))}{8a}$$

and by Maple

$$M_5(a = 2) = 0.1604115273.$$

For $s = 4$ Maple gives

$$M_5(a) = \frac{\pi(-8\sinh(a)\cosh(a)^3 - 4\sinh(a)\cosh(a) + 16\cosh(a)^4 - \sinh(4a) - 4\sinh(2a))}{16a}$$

and by Maple

$$M_5(a = 2) = 0.6034996741.$$

Example 6

$$M_6(a) = \int_{-\infty}^{\infty} \frac{\cos(x)dx}{(x^2 + a^2)^2(x^2 + b^2)} \qquad (a > 0, b > o, a \neq b)$$

and by Maple

$$M_6(a, b) = \frac{\pi[e^{-b}(a^5(b + 1) - a^3b^3 - 5a^3b^2) + e^{-a}(a^3b^3 + 5a^2b^3 - b^5(a + 1))]}{2(a^2 - b^2)^3a^3b^3}$$

and by Maple

$$M_6(a = 2, b = 1) = 0.06740936539.$$

2.3 Examples derived using differentiation under the integral operator

In all cases we have considered the limits of integration as parameters free. Assuming validity we may therefore differentiate with respect to a parameter under the integral.
Example 8

We have using $A = a^2$,

$$\int_{-\infty}^{\infty} \frac{cos(x)dx}{x^2 + A} = \sqrt{a}e^{-\sqrt{A}}/\sqrt{A} \qquad (A > 0)$$

so that

$$\int_{-\infty}^{\infty} \frac{cos(x)dx}{(x^2+A)^2} = \frac{d}{dA}\frac{\sqrt{a}e^{-\sqrt{A}}}{\sqrt{A}}a,$$

and so on for further derivatives.
Example 9

$$\int_{-\infty}^{\infty} \frac{(4\cos^3(x) - 3\cos(x))dx}{x^2+a^2}$$

Note that the numerator in the integral is $\cos(3x)$ so that there should be agreement with

$$\int_{-\infty}^{\infty} \frac{\cos(3x)dx}{x^2+a^2} = 3\int_{-\infty}^{\infty} \frac{\cos(x)dx}{x^2+(3a)^2} = \frac{\pi e^{-3a}}{a} \qquad (a > 0)$$

Note the expression for $\cos(3x)$ is sometimes used in solving a cubic equation. Other integrals with different denominators may come to mind.

2.4 Remarks on residue examples

All the examples we have studied are correctly assumed using the Maple code - a simple check is to use straight forward integration. Note however that examples chosen would appear to be capable of evaluation by residue calculus.

A general question concerns the impact of symbolic code on impact mathematics and perhaps on contemporary classical mathematics. Can there be surprises?

2.5 A difficult example due to Copson

Perhaps the most difficult example given in Copson is on p.153, Miscellaneous examples 15. We quote "Prove that the residue of the function $e^{niz}/(z^2 - 2z\cos\alpha + 1)^2$, $(n > 0, 0 < \alpha < \pi)$, at the pole which lies in the upper half-plane is $-i\lambda e^{ni\cos\alpha}$ where

$$\lambda = \frac{e^{-n\sin\alpha}(n\sin\alpha + 1)}{4\sin^3\alpha}.$$

Hence show that

$$\int_0^{\infty} \frac{x(x^2+1)\sin nx dx}{(x^4 - 2x^2\cos 2\alpha + 1)^2} = \frac{\pi\lambda\sin(n\cos\alpha)}{4\cos\alpha}."$$

There is a single pole with the usual semicircular contour, the residue being simple; but complication arises from the remaining terms in the function.

The Maple code gives a very extensive result with complex variables. For $n = 1$, $\alpha = \pi/4$, and yet the Maple result equals the Copson answer 0.4294659208.

3 Maple code in application to the skewness estimator for the negative binomial distribution

We consider the negative binomial distribution with probability generating function

$$g(t;k,p) = (p+1-pt)^{-k} \qquad (p>0, k>0)$$

following the notation in Bowman and Shenton (2007). The maximum likelihood estimators $(\hat{p}, \hat{k})$ are found from the equation

$$\hat{k}\hat{p} = m_1',$$

$$\frac{n_0}{k} + n_1\left(\frac{1}{k} + \frac{1}{k+1}\right) + n_2\left(\frac{1}{k} + \frac{1}{k+1} + \frac{1}{k+2}\right) + \cdots = N \ln\left(1 + \frac{m_1'}{m}\right)$$

for a random sample $(n_0, n_1, \cdots)$. From Fisher (1922) the variance of the estimator $\hat{k}$ is the series. From our paper (Bowman and Shenton p.100) an element in the impact matrix L_{kp} is

$$i_{kk} = \sum_0^\infty \frac{1}{q^k}\left(\frac{p}{q}\right)^x \frac{\Gamma(k+1)}{\Gamma(k)}\left[\psi_1(k) - \psi(k+x)\right], \quad (q = p+1)$$

ψ_1 being the derivative of the Psi function $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$, then the term in Psi functions is

$$\psi_1(k) - \psi_1(k+x) = \frac{1}{k^2} + \frac{1}{(k+1)^2} + \cdots + \frac{1}{(k+r-1)^2}$$

Fisher expands this in terms of $r = p/(p+1)$ to obtain

$$i_{kk} = \sum_0^\infty \frac{r^x}{x}\frac{(r-1)!\Gamma(k)}{r(k+x)}. \tag{2}$$

Using the Maple code and equation (2) we have checked out the coefficient of r^{10} in (2). It turns out to be the correct coefficient (see Bowman and Shenton, 2007, pp106-107). The Maple code succeeds in giving an expression in the term $r = p/(p+1)$ for

$$\sum_x = 1^\infty \frac{1}{q^k}\frac{r^x}{x!}\frac{\Gamma(k+x)}{\Gamma(k)}\left(\frac{1}{k^s} + \frac{1}{(k+1)^s} + \cdots + \frac{1}{(k+x-1)^s}\right) \quad (k,p>0)$$

when $s = 4$. However it is extremely large in form and occupies several lines of output.

In conclusion it appears that the Maple code succeeds in giving a correct result parametrically, and this parametric series agrees for particular sets of (k, p, s).

4 Conclusion

Symbolic language (Mathematika, Maple, Reduce) started to appear three or more decades ago. Our main experience is with Maple code at Oak Ridge National Laboratory. We use an example briefly (see bowman and Shenton, (2007)). There is the .....

$$F(x) = \sum_{1}^{\infty} \left(\frac{p}{q}\right)^{x} \frac{\Gamma(k+x)}{\Gamma(k)} \left(\frac{1}{k^s} + \frac{1}{(k+1)^s} + \cdots + \frac{1}{(k+r-1)^s}\right)^{m}$$

where $k > 0$,$s = 1, 2, \cdots$;, $m = 1, 2, \cdots$, $q = p + 1$, $0 < p < 1$. For $s = 1$. $m = 1$, the Maple code simplifies it. However for $s = 2$, $m = 3$ the Maple code result expression will be extreme and difficult to interpret.

Appendix 1 Copson's Letter

Harvard University, Cambridge, Mass.
Nov. 12. 1957

Dear Mr. Shenton,

When I wrote my Complex Variable book, I did not attempt to enunciate in the greatest generality but only in a form sufficiently general for my purpose. The subject is hard enough as it is. I therefore restricted myself to poles in the enunciation of the residue theorem - that makes it slightly easier to prove - but I could have allowed instead any form of isolated singularity as Valiron does. His result includes mine.

I didn't know Thron's book - but I think his enunciation is wrong. To take Valiron's form of the enunciation, if there were an infinite number of singular points in **D**, they would have a limit which would be a non-isolated essential singularity and this limit point would either belong to **D** or to its boundary. And a non-isolated essential singularity upsets everything. The fact the Thron says "at most denumerable" doesn't help.

The residue theorem relates to a closed and within and on it there can be most a finite number of isolated singular points.

Yours truly,

E. T. Copson

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