A FASCINATING SERIES DUE TO STIELTJES

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Abstract

Stieltjes in later life wrote an article on a definite integral which could not be evaluated by an analytical continued fraction. Questions of divergency of series arise and these we answer mainly from G. H. Hardy's book on divergency. G. H. Hardy was a pure mathematician and yet here an example really concerns applied mathematics.

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1 INTRODUCTION

Stieltjes (1886) shortly after the publication of his study of semi-convergent (divergent) series produced an intriguing case based on the integral

$$\phi(a) = \int_0^a e^{x^2} dx, \qquad (a > 0)$$

which he says readily results in the divergent series

$$\phi(a) = e^{a^2}\left(\frac{1}{2a} + \frac{1}{4a^3} + \frac{1 \cdot 3}{8a^5} + \frac{1 \cdot 3 \cdot 5}{16a^7} + \cdots\right).$$

The first surprise comes from integration by parts, and from Stieltjes

$$\phi(a) = \frac{e^{a^2}}{2a} + \frac{1}{2}\int_{a_1}^a \left(\frac{e^{x^2}}{x^2}\right) dx;$$

more generally

$$\phi(a) = T_1 + T_2 + \cdots + T_n + R_n,$$

where

$$T_s = \frac{1 \cdot 3 \cdots (2s-1)}{2^s a^{2s-1}} \cdot e^{a^2} \qquad (s = 2, \cdots)$$

$$T_1 = e^{a^2}/(2a),$$

$$R_n = \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \int_{a_n}^a \left(\frac{e^{x^2}}{x^{2n}}\right) dx. \tag{1}$$

An insight into this step (of prime importance) is to see that the problem with divergent series of positive terms is where to make the cut. There will be an n such that

$$\phi(a) > T_1 + T_2 + \cdots + T_{n-1}$$

but

$$\phi(a) < T_1 + T_2 + \cdots + T_{n-1} + T_n.$$

To a certain extent this explains the range (a_n, a) in the remainder R_n. Clearly if $a_n < a$, there will be a positive adjustment; if $a_n > a$, there will be a negative adjustment.

2 THE SECOND SURPRISE

Stieltjes now gives some reason to say that a_n is a root of the equation

$$\frac{1}{(2n-1)a^{2n-1}0!} + \frac{1}{(2n-3)a^{2n-3}1!} + \frac{1}{(2n-5)a^{2n-5}2!} + \cdots = 0$$

This seems to be intuitively obvious from R_n using

$$\frac{e^{x^2}}{x^{2n}} \sim \left(1 + x^2 + \frac{x^4}{2!} + \cdots\right)/x^{2n},$$

along with the fact that α_n is a root of

$$\phi(a) = T_1 + T_2 + \cdots + T_n.$$

Stieltjes now produces the series, in descending powers of n,

$$a_n^2 = n - \frac{1}{6} + \frac{8}{405n} + \frac{68}{25515n^2} - \frac{5582}{3444525n^3} + \cdots$$

We quote his error analysis:

n	a_n^2	Approximation
1	0.85402	0.85413
2	1.84365	1.84367
5	4.8373767	4.8373776

3 REMAINDER

From R_n in (1), using the transformation $x^2 = t$, we have

$$R_n = \frac{1 \cdot 3 \cdots (2n-1)}{2^{n+1}} \int_{a_n^2}^{a^2} \left(\frac{e^t}{t^{n+1/2}}\right) dt,$$

and from $t = n + u$

$$R_n = \frac{1 \cdot 3 \cdots (2n-1)}{2^{n+1} n^{n+1/2}} e^n \int_{a_n^2 - n}^{a^2 - n} e^u \left(1 + \frac{u}{n}\right)^{-n-1/2} du$$

from $t = n + u$.

Stieltjes uses the components

$$A_n = \int_{a_n^2 - n}^{0} e^u \left(1 + \frac{u}{n}\right)^{-n-1/2} du$$

$$B_n = \int_{0}^{a^2 - n} e^u \left(1 + \frac{u}{n}\right)^{-n-1/2} du$$

with asymptotic ($n \to \infty$) developments. For example,

$$A_n = \frac{1}{6} - \frac{13}{1080n} - \frac{353}{90720n^2} + \frac{14223}{1088640n^3} + \cdots$$

4 THE CASE $a = 4$

Here $a_{16}^2 = 16 - 0.018508504$, and $A_{16} < 0.165899471$ and $\sum_{s=1}^{16} T_s = 1,149,400.605339014$, using extended precision on a computer. Stieltjes quotes $\sum_1^{16} T_s = 1,149,400.605$.

Using $T_{17} = 0.176317033$, $A_{16}T_{17} = 0.029250903$ we find $\phi(4) = 1,149,400.634589927$, in agreement with Stieltjes. Clearly Stieltjes must have used more precision than that indicated in his $\sum_1^{16} T_s$; how did he do it?

5 RELATION TO NORMAL INTEGRAL

It is readily seen (Stieltjes, 1886, p63) that

$$\phi(a) = \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-x^2/2} dx}{2a - x^2}$$

and simplifying the Cauchy principal value,

$$\phi(a) = \frac{a}{2\sqrt{\pi}} \int_0^{a^2} \left(\frac{e^t}{\sqrt{a^2 - t}} - \frac{e^{-t}}{\sqrt{a^2 + t}} \right) \frac{dt}{t} - \frac{ae^{-a^2}}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-t} dt}{(t + a^2)\sqrt{t + 2a^2}}$$

which can be evaluated using quadrature. Applications are under consideration.

Here is a caveat concering Stieltjes papers. We have been careful in the account above to keep to the text as set out by Stieltjes. After all the paper was written in 1886 and the so called "New Math" had not appeared at this time. Moreover the Stieltjes paper is ten pages long.

6 DIVERGENT SERIES AND G. H. HARDY

The preface to Hardy (1949) by J. E. Littlewood, one of the top mathematicians at the time - occasionally collaborated with Hardy, and Ramanujin, the Indian mathematical genius. We quote:

PREFACE

Hardy in his thirties held the view that the late years of a mathematician's life were spent most profitably in writing books; I remember a particular conversation about this, and though we never spoke of the matter again it remained an understanding. The level below his best at which a man is prepared to go on working a full stretch is a matter of temperament; Hardy made his decision, and while of course he continued to publish papers his last years were mostly devoted to books; whatever has been lost, mathematical literature has greatly gained. All his books gave him some degree of pleasure, but this one, his last, was his favorite. When embarking on it be told me that he believed in its value (as he well might), and also that he looked forward to the task with enthusiasm. He had actually given lectures on the subject at intervals ever since his return to Cambridge in 1931, and had at one time or another lectured on everything in the book except Chapter XIII.

The title holds curious echoes of the past, and of Hardy's past. Abel wrote in 1828: 'Divergent series are the invention whatsoever.' In the ensuing period of critical revision they were simply rejected. Then came a time when it was found that something after all could be done about them. This is now a matter of course, but in the early years of the century the subject, while in no way mystical or unrigorous, *was* regarded as sensational, and ab an aroma of paradox and audacity.

J. E. LITTLEWOOD

August 1948

We quote from Hardy (1949, p329):

For example, the series,

$$S(n) = \sum \frac{(-1)^{r-1} B_r}{2r!} f^{(2r-1)}(n)$$

let us suppose that $f(x) = \log x$ and $a = 0$, so that $F(x) = x \log x - x$. Then our conditions are satisfied for $r \geq 1$, and we are led to the formulae

$$\log n! = \sum_1^n \log m = (n + \frac{1}{2}) \log n - n + C + \frac{B_1}{1 \cdot 2} \frac{1}{n} - \frac{B_2}{3 \cdot 4} \frac{1}{n^3} + \frac{B_3}{5 \cdot 6} \frac{1}{n^5} - \cdots, \qquad (2)$$

$$C = 1 - \frac{B_1}{1 \cdot 2} + \frac{B_2}{3 \cdot 4} - \frac{B_3}{5 \cdot 6} + \frac{B_4}{7 \cdot 8} - \cdots, \qquad (3)$$

where

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42},$$
$$B_4 = \frac{1}{3}. \quad B_5 = \frac{5}{66}.$$

The series are semi-convergent, and can be used to calculate $\log n!$ and C. We shall see later that $C = \frac{1}{2} \log 2\pi$.

We cannot calculate C with great accuracy from (2) because $n = 1$ is too small. The least term is that last written, which is -.00059 and we can calculate $C = .919 \cdots$, to 3 places, by stopping there. This value of C, used in equation (1), would then give a fairly accurate value for $\log n!$ for large n. On the other hand we could calculate C, with much greater accuracy, by using (1) with a fairly large n and computing $\log n!$ independently.

In practice the C of a given f would be computed by writing

$$\sum f(n) = f(1) + \cdots + f(N) + \sum f(n + N),$$

and applying our formulae to the last series, for which they will be more effective the larger N. A judicious choice of N should then make both parts of the calculation practicable with considerable accuracy.

The method may be applied to convergent series whose convergence is inconveniently slow. In this case we must take $a = \infty$, so that $F'(n) \to 0$, and C is the sum of the series. Thus Euler, taking the $f(x) = (x+9)^{-2}$, calculated

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \cdot + \frac{1}{9^2} + \sum \frac{1}{(n+9)^2}$$

to 18 places of decimals.

For further reading we suggest Brezinski (1980), Stieltjes' (1886, 1905, 1918) works and letters, the books of Wall (1948) and Perron (1957).

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