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THE DERIVATIVE OF A CONTINUED FRACTION

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Abstract

The paper considers second order continued fractions associated with (I) the Psi function $\psi(z)$, (II) the continued fraction component in $\ln\Gamma(z)$ due to Stieltjes. The second order sequences k_s^*/k_s provide approximants, some of which are remarkably close. In addition a series form for the convergent χ_s/ω_s associated with a continued fraction provides an expression for the derivatives of a continued fraction. The implementation uses a Maple code for derivatives.

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1 Introduction

We have discussed this subject in our paper Shenton and Bowman (2005, p.21). For a continued fraction it is assumed that

$$\frac{b_1}{z+}\ \frac{b_2}{1+}\ \frac{b_3}{z+}\ \frac{b_4}{1+}\ \cdots = \int_0^\infty \frac{d\sigma(u)}{u+z} \quad (\Re(z) > 0)$$

the integral of Stieltjes form, the partial numerators being real and positive. The notation for the continued fraction is

$$\cfrac{b_1}{z + \cfrac{b_2}{z + \cfrac{b_3}{z + \cfrac{b_4}{z + \cdots}}}}.$$

In general the partial numerators are positive reals.

In this paper we introduce new examples of second order continued fractions. We are given a series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_3}{z^2} - \frac{c_4}{z^4} + \cdots$$

which in some domain may be convergent or divergent. A good reference is given in Borel (1928); see also Bowman and Shenton (1989).

To high light the basic procedures we give a simple example. For the Psi function $\psi(\cdot)$ we have the continued fraction expression

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{a_0}{z^2+}\ \frac{a_1}{1+}\ \frac{a_2}{z^2+}\ \cdots \quad (\Re(z) > 0) \tag{1}$$

where

$$a_0 = \frac{1}{12},\quad a_1 = \frac{1}{10},\quad a_2 = \frac{79}{210},\quad a_3 = \frac{1205}{1659},\quad a_4 = \frac{262445}{209429},\quad a_5 = \frac{2643428417511}{1429053441530}.$$

There are other interesting forms of (1).

(a) Integral form

$$\psi(z) = \ln z - \frac{1}{2z} - \int_0^\infty \frac{du}{(z^2+u)(e^{2\pi\sqrt{u}}-1)}$$

given by Shenton and Bowman (1971, p.552). The integral is Stieltjes and subscribes to the form

$$\int_0^\infty \frac{d\sigma(u)}{u+z^2}$$

which arises for a class of continued fraction (Wall, 1948). It suggests a continued fraction

$$\frac{d_1}{z^2+}\,\frac{d_2}{1+}\,\frac{d_3}{z^2+}\,\frac{d_4}{1+}\,\cdots.$$

(b) Asymptotic series

$$\psi(z) = \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}} \qquad (z \to \infty \text{ in } [\arg] < \pi)$$

and in more detail

$$\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{20z^4} - \frac{1}{252z^6} + \frac{1}{210z^8} - \frac{1}{12z^{10}} \cdots,$$

this series, extended, may be used to set up the continued fraction using Wall.s algorithm (Wall, 1948, p.194).

We now return to expression (1) for $\psi(z)$ and differentiate with respect to z, finding for the derivative of $\psi(z)$

$$\psi_1(z) = \frac{1}{z} + \frac{1}{2z^2} - \frac{d}{dz}\left(\frac{a_0}{z^2+}\,\frac{a_1}{1+}\,\frac{a_2}{z^2+}\,\frac{a_3}{1+}\,\cdots\right).$$

Thus

$$\psi_1(z) - \frac{1}{z} - \frac{1}{2z^2} = -\frac{d}{dz}\left(\frac{a_0}{z^2+}\,\frac{a_1}{1+}\,\frac{a_2}{z^2+}\,\frac{a_3}{1+}\,\cdots\right)$$

so that the second order continued fraction

$$\frac{d}{d\omega}\left(\frac{a_0}{\omega+}\,\frac{a_1}{1+}\,\frac{a_2}{\omega+}\,\frac{a_3}{1+}\,\cdots\right)$$

has convergent $k_s^*(\omega)/k_s(\omega)$, $s = 1, 2, \cdots$, to the function $F(z)$ where

$$F(z) = \frac{1}{2z}\left\{\psi_1(z) - \frac{1}{z} - \frac{1}{2z^2}\right\}.$$

We shall define $F(z)$ to be the limiting value of the 2nd order continued function sequence k_s^*/k_s. The algorithm of deriving k_s^*/k_s is given in Shenton (1957), and Shenton and Bowman (2005).

For examples, if $z = 1$, then $k_s^*(1)/k_s(1)$, $s = 1, 2, \cdots$. converges to $\frac{1}{2}\left(\frac{\pi^2}{6} - \frac{3}{2}\right) = 0.072$. Moreover $k_0^* = 0$, $k_1^* = k_2^* = 1/12$. $k_1 = 1$, $k_2 = 1 + \frac{1}{10}\left(2 + \frac{1}{10} + \frac{79}{210}\right)$. The approximants are $k_1^*/k_1 = 0.08333\cdots$, $k_2^*/k_2 = 0.067$.

Table 1 Approximants for $\psi_1(1)$, values of k_s^*/k_s

s	1	2	3	4	5	converges to
k_s^*/k_s	0.08333	0.06679	0.07492	0.07088	0.07336	$\frac{1}{2}\left(\frac{\pi^2}{6}-\frac{3}{2}\right)$ or 0.07247
$\psi_1(1)$	1.66667	1.35889	1.64984	1.64175	1.64671	1.64493

In the table the odd convergents corresponding to 1, 3, 5, form a monotonic decreasing sequence; similarly the even convergents, corresponding to 2, 4 form a monotonic increasing sequence. Altogether the approximations are very satisfactory.

2 Examples

Example 1

In Shenton and Bowman (1971) we have given continued fractions for derivatives of the Psi functions $\psi_m(z)$, $m = 0, 1, 2, \cdots$, with $\psi(z) = d \ln \Gamma(z)/dz$. For $\psi_m(z)$ itself the first those partial numerators $C_1^{(m)}$, $C_2^{(m)}$, $C_3^{(m)}$ we stated explicitly. Looking at the continued fractions listed, it turns out that only one, that of $\psi_2(z)$ has all partial numerators defined. Thus the continued fraction is

$$\frac{1}{z^2+}\;\frac{p_1}{1+}\;\frac{q_1}{z^2+}\;\frac{p_2}{1+}\;\frac{q_2}{z^2+}\;\cdots$$

where

$$p_s = \frac{s^2(s+1)}{4s+2}, \quad q_s = \frac{s(s+1)^2}{4s+2}.$$

The example is due to Stieltjes (1918, p.388) An obvious question is why $\psi_2(z)$ and its continued fraction? ‘

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^2}\left\{\frac{1}{z^2+}\;\frac{p_1}{1+}\;\frac{q_1}{z^2+}\;\frac{p_2}{1+}\;\frac{q_2}{z^2+}\cdots\right\} \quad (\Re(z) > 0) \tag{2}$$

(Note: There is a typo error in Shenton and Bowman (1971, p.548, (7b)); the first term on the right should be negative).

From (2),

$$\left(\psi_2(z) + \frac{1}{z^2} + \frac{1}{z^3}\right)2z^2 = -\frac{1}{z^2+}\;\frac{p_1}{1+}\;\frac{q_1}{z^2+}\;\cdots$$

differentiate with respect to z

$$2\frac{d}{dz}\left(z^2\psi_2(z) + 1 + \frac{1}{z}\right) = 2z\frac{d}{d\omega}\left(\frac{1}{\omega+}\;\frac{p_1}{1+}\;\frac{q_1}{\omega+}\;\frac{p_2}{1+}\;\frac{q_2}{\omega+}\cdots\right) \quad (\omega = z^2)$$

resulting

$$\left(2\psi_2(z) + z\psi_3(z) - \frac{1}{z^3}\right) = \lim \frac{k_s^*(\omega)}{k_s(\omega)} \quad (\omega = z^2)$$

and

$$\frac{k_1^*}{k_1} = \frac{1}{\omega^2}, \quad \frac{k_2^*}{k_2} = \frac{1}{\omega^2 + p_1(2\omega + q_1 + p_1)},$$

the first should be an upper bound and the second is a lower bound. Examples are given in Table 2.

Table 2 Values of k_s^*/k_s of example 1

z	1	2	3	4	5	converges to
1/2	16.0000000	1.7777778	12.6137566	3.0634024	10.9613493	7.046952
1	1.0000000	0.5000000	0.7916666	0.7322530	0.6134259	0.685711
2	0.0625000	0.0526316	0.0552326	0.0544314	0.0547382	0.054651

Example 2

$$\ln\Gamma(z) = \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \frac{a_0}{z+}\,\frac{a_1}{z+}\,\frac{a_2}{z+}\cdots \qquad (\Re(z) > 0)$$

where a_0 to a_6, are given in Wall (1948).

The continued fraction may be written

$$\frac{\alpha_1 z}{z^2+}\,\frac{\alpha_2}{1+}\,\frac{\alpha_3}{z^2+}\,\frac{\alpha_4}{1+}\,\frac{\alpha_5}{z^2+}\,\frac{\alpha_6}{1+}\cdots(\Re(z) > 0),$$

so we write

$$\frac{1}{z}\left\{\ln\Gamma(z) - \left(z - \frac{1}{2}\right)\ln z + z - \frac{\ln(2\pi)}{2}\right\} = \frac{\alpha_1}{\omega+}\,\frac{\alpha_2}{1+}\,\frac{\alpha_3}{\omega+}\,\frac{\alpha_4}{1+}\cdots \qquad (\omega = z^2)$$

and the right hand side is in the correct form to agree with the 2nd order continued fraction. Differentiate with respect to z, we have

$$\frac{d}{dz}\frac{1}{z}\left\{\ln\Gamma(z) - \left(z - \frac{1}{2}\right)\ln z + z - \frac{\ln(2\pi)}{2}\right\} = 2z\frac{d}{d\omega}\left\{\frac{\alpha_1}{\omega+}\,\frac{\alpha_2}{1+}\,\frac{\alpha_3}{\omega+}\,\frac{\alpha_4}{1+}\cdots\right\}.$$

Allowing for a change in sign when we consider derivatives, the sequences k_s^*/k_s are approximants to

$$-\frac{1}{2z}\frac{d}{dz}\left\{\frac{\ln\Gamma(z)}{z} - \left(1 - \frac{1}{2z}\right)\ln z + 1 - \frac{\ln(2\pi)}{2z}\right\}$$

$$= -\frac{1}{2z}\left\{\frac{\psi_1(z)}{z} - \frac{\ln\Gamma(z)}{z^2} - \frac{\ln(z)}{2z^2} - \left(\frac{1}{z} - \frac{1}{2z^2}\right) + \frac{\ln(2\pi)}{2z^2}\right\}$$

with

$$k_1 * = 1/12 = k_2^*, \quad k_1 = \omega^2, \quad k_2 = \omega^2 + \alpha_2(2\omega + \alpha_3 + \alpha_2).$$

and when $z = 1/2$ becomes

$$-\left\{2\psi_1(\frac{1}{2}) + 4\ln 2\right\} = 2\gamma$$

Take $z = 1$. The 2nd order continued fraction (k_s^*/k_s) converges to the function

$$F(1) = -\frac{1}{2}\left\{\psi(1) - \left(1 - \frac{1}{2}\right) - \frac{\ln(2\pi)}{2}\right\} = 0.07908$$

Table 3 Values of k_s^*/k_s of example 2

z	1	2	3	4	5	converges to
1/2	1.3333333	0.9395973	1.2501182	1.0464910	1.1833200	2γ
1	0.0833333	0.0774363	0.0797778	0.0787481	0.0792193	0.07908
2	0.0052083	0.0051200	0.0051306	0.0051288	0.0051292	0.0051291

3 A determinantal reduction formula for continued fractions

Consider the continued fraction $\frac{c_0}{z+}\,\frac{c_1}{z+}\,\frac{c_2}{z+}\,\cdots$ with convergents $\frac{\chi_s}{\omega_s}$, $s = 1, 2, \cdots$. Then, for example,

$$\chi_1 = c_0,\quad \chi_2 = zc_1,$$
$$\omega_1 = z,\quad \omega_2 = z + c_1.\quad (\chi_0 = 0,\ \omega_0 = 1).$$

We may therefore consider the determinant

$$\begin{vmatrix} \chi_s & \chi_{s+1} \\ \omega_s & \omega_{s+1} \end{vmatrix}$$

using the recurrence formulas. Expanding the determinant leads to

$$\frac{\chi_s}{\omega_s} = \frac{\chi_0}{\omega_0} + \left(\frac{\chi_1}{\omega_1} - \frac{\chi_0}{\omega_0}\right) + \cdots + \left(\frac{\chi_s}{\omega_s} - \frac{\chi_{s-1}}{\omega_{s-1}}\right) + \cdots +$$

at least formally. Hence

$$\frac{\chi_s}{\omega_s} = \frac{c_0}{\omega_0\omega_1} + \frac{c_0c_1}{\omega_1\omega_2} + \frac{c_0c_1c_2}{\omega_2\omega_3}. \tag{3}$$

Example 3

The Laplace continued fraction is

$$\frac{1}{z+}\,\frac{1}{z+}\,\frac{2}{z+}\,\frac{3}{z+}\,\cdots \qquad (z>0)$$

so $\omega_1 = z$, $\omega_2 = z^2 + z$, $\omega_3 = z^3 + 3z$, $\omega_4 = z^4 + 6z^2 + 3$. A simple formula is $\omega_s = e^{-\frac{1}{2}d_x^2}x^2$ the polynomial being Hermite.

Now using (3) we may set up an expression for derivatives of a continued fraction using the Maple symbolic code.

4 Conclusion

We now pay attention to the function $F(z)$ which represents the value to which the 2nd order continued fractions converges. If z is real and positive is $F(z)$ positive? For the first example we knew (Shenton and Bowman, 1971)

$$\begin{aligned} F(z) &= \frac{1}{2z}\left\{\psi_1(z) - \frac{1}{z} - \frac{1}{2z^2}\right\} \\ &= \frac{1}{2z}\frac{2\pi}{3}\int_0^\infty \frac{y\sqrt{x}dx}{xz^2(y-1)^2} \qquad (y = e^{2\pi\sqrt{x}}) \end{aligned}$$

which is positive for $z > 0$.

Now consider the expression for $F(z)$ arising from the continued fraction from $\ln\Gamma(z)$. In the sequel for this case

$$\begin{aligned} F(z) &= -\frac{1}{2z}\frac{d}{dz}\left\{\frac{\ln\Gamma(z)}{z} - \left(1 - \frac{1}{2z}\right)\ln z + 1 - \frac{\ln(2\pi)}{2z}\right\} \\ &= -\frac{1}{2z}\left\{\frac{\psi(z)}{z} - \frac{\ln\Gamma(z)}{z^2} - \frac{\ln z}{2z^2} - \left(\frac{1}{z} - \frac{1}{2z^2}\right) - \frac{\ln(2\pi)}{2z^2}\right\} \end{aligned}$$

We have been unable to prove this is positive, for z real and positive. So we test out several cases.

Case with $z = \frac{1}{2}$:

$$F(z) = \{-2\psi(2) - 2\ln\pi - 2\ln\frac{1}{2} + 2\ln(2\pi)\} = 2\gamma \ \ (\gamma = \text{Euler's constant}).$$

The upper bound is 4/3. It is quite remarkable that in the Handbook of Mathematical Functions.... section 6.3.3, we have the single entry (also see "dlmf.nist.gov", 5.4.13)

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2 = 1.9635100260....$$

As far as we can tell it is not referenced. What a remarkable piece of luck?

Now take $F(1)$: We have

$$F(1) = -\frac{1}{2}\left\{-\gamma - \frac{1}{2} + \frac{\ln(2\pi)}{2}\right\} = 0.0058.$$

The upper bound is 1/12.

For $z = 2$,

$$\begin{aligned} F(2) &= -\frac{1}{4}\left\{-\frac{\psi(2)}{2} - \frac{\ln 2}{8} - \frac{3}{8} + \frac{\ln(2\pi)}{8}\right\} \\ &= -\frac{1}{4}\left\{\frac{1}{2}(1-\gamma) - \frac{3}{8} + \frac{\ln(2\pi)}{8}\right\} = 0.0051\cdots. \end{aligned}$$

The upper bound is 0.0208.

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