*Far East J.Appl.Math.* 32(1) (2008), 1-11

# MONTAGE FOR THE GAMMA FUNCTION

L.R. SHENTON[1] and K.O. BOWMAN[2]

[1]Department of Statistics, University of Georgia
Athens, Georgia 30602, U.S.A.
[2]Computational Sciences and Engineering Division
Oak Ridge National Laboratory, P.O.Box 2008, 4500N, MS-6191
Oak Ridge, TN 37831-6191, U.S.A.
e-mail: bowmanko@ornl.gov

March 26, 2008

## Abstract

The $\ln\Gamma(z)$ consists of three parts, (i) $(z-\frac{1}{2})\ln z - z$, (ii) the constant $\ln\sqrt{2\pi}$, and (iii) the Stieltjes continued fraction $J(z)$. The partial numerators for $J(z)$ have been found by Char (*Mathematics of Computation*, 34(150), 1980) and asymptotic forms are needed, along with a conjecture of Stieltjes. Sequences of approximants are set up for $\ln\sqrt{2\pi}$. In another direction we use a second order continued fraction for the exponential function $e^z$, noting that $\frac{d}{dz}e^z = e^z$ so that a derivative of a continued fraction is involved.

2000 Mathematics Subject Classification: 62E20.
*Key words and phrases:* Psi-function.

## 1 Clemshaw's formula (1954)

In the *Handbook of Mathematical Functions*, (4.2.40) we have the continued fraction formulas

$$e^z = \frac{1}{1-}\,\frac{z}{1+}\,\frac{z}{2-}\,\frac{z}{3+}\,\frac{z}{2-}\,\frac{z}{5+}\,\frac{z}{2-}\cdots \qquad (|z|<\infty) \tag{1}$$

$$= 1+\frac{1}{1-}\,\frac{z}{2+}\,\frac{z}{3-}\,\frac{z}{2+}\,\frac{z}{5-}\,\frac{z}{2+}\,\frac{z}{7-}\cdots \qquad (|z|<\infty) \tag{2}$$

$$= 1+\frac{z}{(1-z/2)+}\,\frac{z^2/4\cdot 3}{1+}\,\frac{z^2/4\cdot 15}{1+}\cdots\frac{z^2/4(4n^2-1)}{1+}\cdots \qquad (|z|<\infty)$$

attributed to Clemshaw (1954). By rearrangement of (2) we have the form

$$e^z = 1+\frac{z}{1-z/2+}\,\frac{z^2/4\cdot 3}{1+}\,\frac{z^2/4\cdot 15}{1+}\,\frac{z^2/4\cdot 35}{1+}\cdots$$

leading to

$$\frac{z}{e^z-1} = 1-\frac{z}{2}+\frac{z}{1-\frac{z}{2}+\frac{z^2/(4\cdot 3)}{1+}\,\frac{z^2/(4\cdot 15)}{1+}\,\frac{z^2/(4\cdot 35)}{1+}\cdots} \qquad (|z|<\infty)$$

in which also

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty}\frac{B_n z^n}{n!} \qquad (|z|<2\pi).$$

with Bernoulli numbers $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$. The relationship seems to have been overlooked.

Wall (1948, p348) derived expression (1) for $e^z$ using a more general function namely $\Phi(1, c; z)$. Wall therefore verifies the series for $e^z$ given by Clemshaw, using a completely different approach.

## 2 The continued fraction component in $\ln\Gamma(z)$ and a study of Char

Basically

$$\ln\Gamma(z) = (z-\frac{1}{2})\ln z - z + \frac{1}{2}\ln(2\pi) + J(z) \qquad (z\to\infty \text{ in } \arg|z|<\pi).$$

From which Stieltjes (letter 172, 1905), (1918) gives the partial numerator components $a_1$, $a_2$, $\cdots$, $a_5$ as

$$a_1 = \frac{1}{12},\quad a_2 = \frac{1}{30},\quad a_3 = \frac{53}{210},\quad a_4 = \frac{195}{371},\quad a_5 = \frac{22999}{22737},$$

in the continued fraction

$$J(z) = \frac{a_1}{z+}\,\frac{a_2}{z+}\,\cdots \qquad (\Re(z)>0).$$

Wall (1948, p364) gave two additional terms $a_6$ and $a_7$.

$$a_6 = \frac{29944523}{19733142}, \quad a_7 = \frac{109535241009}{48264275462}.$$

Now Char (1980), using a symbolic program to set up partial numerators in $J(z)$ gave terms to $a_{40}$ (Table 1).

Table 1. Char's Stieljes c.f. for the Gamma function

| $s$ | $a_s$ | $s$ | $a_s$ |
|---|---|---|---|
| 0 | 0.083333 | 16 | 16.053551 |
| 1 | 0.033333 | 20 | 25.065847 |
| 2 | 0.252381 | 24 | 36.077689 |
| 3 | 0.525606 | 28 | 49.089255 |
| 4 | 1.011523 | 32 | 64.100423 |
| 8 | 4.026887 | 36 | 81.111403 |
| 12 | 9.040660 | 40 | 100.122178 |

Studying the coefficients it is clear that there is a regular pattern of $a_s$ ($s = 10$ to 40) and we point-out that the approximation is $a_s/s^2 \sim 1/16 \ \ (s \to \infty)$.

In (letter 173, p.354) Stieltjes (1905) states exactly this result as a conjecture that $a_s/s^2 \sim 1/16$. A result apparently overlooked in the literature. Stieltjes (letter 173, p352-353; 1905) noted the arduous task of setting up $a_1$, $a_2$, $\cdots$, $a_5$, in $J(z)$, realty in, as later turned out, that $a_s \sim s^2/16$. Stieltjes had in mind the continued fraction in

$$\psi\left(\frac{1+a}{2}\right) - \psi\left(\frac{a+1}{4}\right) - \ln(2) = \frac{1}{a+}\,\frac{1^2}{a-}\,\frac{2^2}{a+}\,\frac{3^2}{a-}\,\cdots.$$

which we refer to in the sequel.

# 3 Approximants for $\ln\sqrt{2\pi}$

From (2) we have for the constant $\ln\sqrt{2\pi}$,

$$\ln\sqrt{2\pi}\left\{\ln\Gamma(z) - \left(z - \frac{1}{2}\right)\ln z + z\right\} + J(z) \qquad (\Re(z) > 0)$$

where $J(z)$ has partial numerators $a_1$, $a_2$,$\cdots$.

Three illustrative examples are given in Table 2; error terms are given in Table 3.

Table 2. Approximants to $\ln\sqrt{2\pi}$

| Terms | $z=1/2$ | $z=1$ | $z=10$ |
|---|---|---|---|
| 1 | 0.905698 | 0.91666666 | 0.918935763304702280 |
| 2 | 0.925306 | 0.91935484 | 0.918938540156862671 |
| 3 | 0.916069 | 0.91882716 | 0.918938533168583255 |
| 4 | 0.920932 | 0.91898387 | 0.918938533517609271 |
| 5 | 0.917737 | 0.91891874 | 0.918938533514287047 |
| 6 | 0.919895 | 0.91894935 | 0.918938533204672871 |
| 7 | 0.918284 | 0.91893264 | 0.918938533204672754 |
| $\ln\sqrt{2\pi}$ | | | 0.918938533204672742 |

Table 3. Errors(%) for $\ln\sqrt{2\pi}$

| Terms | $z=1/2$ | $z=1$ | $z=10$ |
|---|---|---|---|
| 1 | 1.440821 | 0.24722726 | 0.0003014239 |
| 2 | -0.692928 | -0.04530287 | -0.0000007565 |
| 3 | 0.312305 | 0.01211971 | 0.0000000039 |
| 4 | -0.216955 | -0.00493357 | -0.0000000000 |
| 5 | 0.130791 | 0.00215358 | 0.0000000000 |
| 6 | -0.104113 | -0.00117745 | -0.0000000000 |
| 7 | 0.071257 | 0.00064140 | 0.0000000000 |

Comments. Approximants are enveloping and improve as $x$ gets larger.

There is a special case when $x = 1/2$. Here,

$$\Gamma\left(\frac{1}{2}\right) = 0 - \frac{1}{2} + \ln\sqrt{2\pi} + \frac{a_1}{\frac{1}{2}-}\ \frac{a_2}{\frac{1}{2}-}\ \frac{a_3}{\frac{1}{2}-}\cdots.$$

where $\Gamma(1/2) = \sqrt{\pi}$. Hence the term in $\sqrt{\pi}$ disappears, leading to

$$\ln 2 = 1 - \frac{2a_1}{\frac{1}{2}+}\ \frac{a_2}{\frac{1}{2}+}\ \frac{a_3}{\frac{1}{2}+}\ \cdots.$$

Thus $\ln 2 < 1$, and $\ln 2 > 1 > 2/3$.

# 4 The derivative of $e^z$ and the corresponding 2nd order c.fs

From

$$e^z = 1 + \frac{z}{1-\frac{z}{2}+}\ \frac{z^2/(4\cdot 3)}{1+}\ \frac{z^2/(4\cdot 15)}{1+}\ \frac{z^2/(4\cdot 35)}{1+}\ , \qquad (z>0)$$

differentiation with respect to $z$ and simplifying, we have

$$e^z = \frac{e^z-1}{z} + \frac{1}{2z}(e^z-1)^2 - \frac{(e^z-1)^2}{z}\frac{d}{dz}C(z^2), \tag{3}$$

where $C(z^2) = \frac{z^2/(4\cdot 3)}{1+} \frac{z^2/(4\cdot 15)}{1+} \cdots$. In $C(z^2)$, replace $z^2$ by $1/\omega$ and carry out the equivalence transformation, leading to

$$e^z = \frac{e^z - 1}{z} + \frac{1}{2z}(e^z - 1)^2 - \frac{2}{z^4}(e^z - 1)^2 \frac{d}{dz} C(\omega),$$

$$C(\omega) = \frac{1/(4\cdot 3)}{\omega+} \frac{1/(4\cdot 15)}{1+} \frac{1/(4\cdot 35)}{\omega+} \frac{1/(4\cdot 63)}{1+}$$

the partial numerator being $1/[4(4s^2 - 1)]$, the partial denominator $\omega$, 1, $\omega$ 1, etc, where $\omega = 1/z^2$. A Stieltjes type continued fractions.

Now the denominator of a Stieltjes type c.f. has given in Shenton and Bowman (2005). If

$$F(z) = \int_0^\infty \frac{d\sigma(t)}{t+z} = \frac{b_1}{z+} \frac{b_2}{1+} \frac{b_3}{z+} \frac{b_4}{1+} \cdots,$$

the integral being a Stieltjes transform, $b_s > 0$, $s = 1, 2, \cdots$, then

$$l.i.s. \frac{k_{2s}^*}{k_{2s}} = \lim \frac{k_s^*}{k_s} = \frac{F(z_2) - F(z_1)}{z_1 - z_2},$$

- $k_s^*$ and $k_s$ follow, for $s = 2, 3, \cdots$,
  $W_{2s-1} = z_1 z_2 W_{2s-2} + \alpha_{2s-1} W_{2s-3} - \beta_{2s-1} W_{2s-5} - z_1 z_2 \gamma_{2s-1} W_{2s-6} + \delta_{2s-1} W_{2s-7}$,
  $W_{2s} = W_{2s-1} + \alpha_{2s} W_{2s-2} - \beta_{2s} W_{2s-4} - \gamma_{2s} W_{2s-5} + \delta_{2s} W_{2s-6}$.

- $k_0^* = 0$, $k_1^* = k_2^* = b_1$, $k_s^* = 0$, $s < 0$,
  $k_0 = 1$, $k_1 = z_1 z_2$, $k_2 = z_1 z_2 + b_2(z_1 + z_2 + b_3 + b_2)$, $k_s = 0$, $s < 0$,

- $\alpha_s = b_s(z_1 + z_2 + b_{s+1} + b_s + b_{s-1})$, $\beta_s = b_s b_{s-2} \alpha_{s-1}$,
  $\gamma_s = b_s b_{s-1} b_{s-2} b_{s-3}$, $\delta_s = b_s b_{s-1} b_{s-2}^2 b_{s-3} b_{s-4}$;

- $(t + z_1)(t + z_2) > 0$ for $x \geq 0$.

If $z_1 = z_2$ then we are considering the derivative at $z = z_1$ or the derivative of a continued fraction. A sequence of lower, and upper bounds will be found.
Example: $z_1 = z_2 = 1$, $F(z) = e$, $b_s = 1/[4(4s^2 - 1)]$. The first approximant is

$$k_0^* = 0, \quad k_1^* = b_1 = 1/12, \quad k_0 = 1, \quad k_1 = 1.$$

Approximant is

$$e - 1 + \frac{1}{2}(e-1)^2 - \frac{2}{12}(e-1)^2 = 2.702445975,$$

and is less than $e$. The second approximant is

$$k_2^* = b_1, \quad k_2 = 1 + \frac{1}{60}\left(2 + \frac{1}{60} + \frac{1}{140}\right)$$

leading to

$$e \sim (e-1) + \frac{1}{2}(e-1)^2 - \frac{2}{12}\frac{(e-1)^2}{1+\frac{1}{30}+\frac{1}{3660}+\frac{1}{8400}} = 2.718500142$$

which as expected, is grater than $e$ (see table 4).

Table 4 Approximants to $\frac{d}{dz}(e^z)|_z = 1$ using c.f.

| | Approx. | Approx.-$e$ |
|---|---|---|
| 1 | 2.70244597579658 | -0.01583585266247 |
| 2 | 2.71850239662646 | 0.00022056816741 |
| 3 | 2.71828053050483 | -0.00000129795422 |
| 4 | 2.71828183335584 | 0.00000000489679 |
| 5 | 2.71828182844768 | -0.00000000001137 |
| $F$ | 2.71828182845905 | |

Also note the simple bounds implied in equation (1). For $z$ real,

$$\begin{aligned} e^z &\le 1 + \frac{z}{1-\frac{z}{2}+\frac{z^2}{12}}, \\ &\ge 1 + \frac{z}{1-\frac{z}{2}+\frac{z^2/12}{1+z^2/60}}. \end{aligned}$$

For $z = 1$, the approximants to $e$ are 19/7 and 193/71; for approximant to $1/e$ the approximants are 7/19 and 71/193.

We note that there is a generalization to the notation that $e^z \cdot e^{-z} = 1$. Euler(1813) gave what is equivalent to

$$\frac{1-\left(\frac{1-z}{1+z}\right)^k}{1+\left(\frac{1-z}{1+z}\right)^k} = \frac{kz}{1+\frac{(k^2-1)z^2}{3+\frac{(k^2-4)z^2}{5+\frac{(k^2-9)z^2}{7+\cdots}}}}.$$

Valid in the plane split from -1 to $-\infty$, and +1 to $+\infty$.

For another example consider

$$\begin{aligned} \psi\left(\frac{a+1}{2}\right) - \psi\left(\frac{a+1}{4}\right) - \ln 2 &= \frac{1}{a-}\,\frac{1^2}{a-}\,\frac{2^2}{a-}\,\frac{3^2}{a-}\,\cdots \qquad (a>0) \\ &= C(a) \ \text{say}. \end{aligned}$$

Note that the partial numerators are 1, $1^2$, $2^2$, $\cdots$, similar to those in $J(z)$, in the latter when $s$ in $a_s$ is large.

Using equivalence transformations,

$$\begin{aligned} C(a) &= a\left\{\frac{1}{A+}\,\frac{1^2}{1+}\,\frac{2^2}{A+}\,\frac{3^2}{1+}\,\frac{4^2}{A+}\,\cdots\right\} \\ &= aC^*(A). \qquad (A = a^2) \end{aligned}$$

We look for an expression for

$$\frac{d}{da}C^*(A) = 2a\frac{d}{dA}C^*(A),$$

and the corresponding 2nd order continued fraction. By differentiation

$$\frac{1}{2}\psi_1\left(\frac{a+1}{2}\right) - \frac{1}{4}\psi_1\left(\frac{a+1}{4}\right) = \frac{C(a)}{a} + a(2a)\frac{d}{dA}C^*(A).$$

The last term on the right is approximated by $-k_s^*/k_s$, where $k_0^* = 0$, $k_1^* = k_2^* = 2a^2$, $k_0 = 1$, $k_1 = (a^2)^2 = a^4$, $k_2 = a^4 + (2a^2 + 5)$. Since $b_1 = 1$, $b_2 = 1^2$, $b_3 = 2^2$ ($b$, numerators).

For simple particular case take $a = 3$ and use the identities,

$$\psi(1) = -\gamma, \qquad (\gamma \text{ is Euler's constant}$$
$$\psi(2) = 1 - \gamma$$

so that $\psi(2) - \psi(1) = 1$, $\psi_1(1) = \pi^2/6$, and $\psi_1(2) = \pi^2/6 - 1$. We are approximating

$$\lambda = \frac{5}{6} - \frac{\ln 2}{3} - \frac{\pi^2}{24} = 0.19105,$$

to second order continued fraction approximants, comparisons are given in Table 5.

Table 5. The derivatives of $\frac{1}{a+}\,\frac{1^2}{a+}\,\frac{2^2}{a+}\cdots$ when $a = 3$

| $s$ | $k_s^*/k_s$ | $s$ | $k_s^*/k_s$ |
|---|---|---|---|
| 2 | 0.2222 | 3 | 0.1731 |
| 4 | 0.2042 | 4 | 0.1762 |
| 6 | 0.1970 | 7 | 0.1779 |
| 8 | 0.1933 | 9 | 0.1791 |
| 10 | 0.1912 | 10 | 0.1799 |

Note that the approximants are enveloping. Computations in Table 5 are based on a simplified version of the recurrence scheme described in section 4. Briefly

$$k_s = b_{s+1}b_s\cdots b_1K_s, \qquad k_s^* = b_{s+1}b_s\cdots b_1K_s^* \qquad (s = 0, 1, \cdots)$$

Let

$$z_1 + z_2 + b_{s+1} + b_s + b_{s-1} = C_s, \qquad (s \geq 3),$$
$$z_1 + z_2 + b_3 + b_2 = C_2.$$

Recurrence scheme,

$$b_{2s}w_{2s-1} = z_1z_2w_{2s-2} + C_{2s-1}w_{2s-3} - C_{2s-2}w_{2s-5} - z_1z_2w_{2s-6} + b_{2s-3}w_{2s-7},$$
$$b_{2s+1}w_{2s} = w_{2s-1} + C_{2s}w_{2s-2} - C_{2s-1}w_{2s-4} - w_{2s-5} + b_{2s-2}w_{2s-6} \quad (s = 2, 3, \cdots).$$

## 5 Concluding remark on the asymptotic $s^2/16$, for $a_s$ in $J(z)$

Stieltjes had worked out terms in the continued fraction for $J(z)$ up to and including $a_5$. Note then, that

Table 6. Continued fraction $J(z)$

| $s$ | $a_a$ | $s^2/16$ |
|---|---|---|
| 3 | 0.526 | 0.562 |
| 4 | 1.011 | 1.000 |
| 5 | 1.517 | 1.516 |
| 6 | 2.270 | 2.250 |

Did Stieltjes project the asymptote from these cases. There may be clues in his letter 177. For the case

$$\frac{a_0}{x} - \frac{a_1}{x^2} + \frac{a_2}{x^3} + \cdots = \frac{a_0}{x+}\ \frac{p_1}{x+}\ \frac{q_1}{x+}\ \frac{p_2}{x+}\ \frac{q_2}{x+} \cdots$$

he gives, using diagonal notation in the determinants

$$p_n = \frac{A_{n-1}B_n}{A_nB_{n-1}}, \qquad q_n = \frac{A_{n-1}B_{n-1}}{A_nB_n}$$

with $A_0 = B_0 = 1$ and the persymetric quotient determinants

$$A_n = |a_0, a_2, \cdots, a_{2n-2}|, \qquad B_n = |a_1, a_3, \cdots, a_{2n-1}|.$$

The underlying series for our case $J(z)$ is

$$J(z) = \sum_{pq}^{\infty} \frac{B_{2p}}{2p(2p-1)} \frac{1}{z^{2p-1}}.$$

Using Bernoulli numbers $B_2 = 1/6$, $B_3 = 0$, $B_4 = 1/30$. Stieltjes (letter 177, 1905) gives example. The big problem relates to $A_n$and $B_n$, and their quotient, each being persymetric.

## References


Char. B.W. (1980). On Stieltjes' continued fraction for the Gamma functions, *Mathematics of Computation*, 34(150), 574-551.

Clemshaw, C. W. (1954). Polynomial Approximations to Elementary Functions, *Math. Tables Aids Comp.*, 8, 143-147.

Euler, L. (1813). "Commentatio in fractionem continuam qua illustris La Grange potestates binomiales expressit," *Mémoires de l'Académie imperiale des sciences de St. Pétersbourg*, vol.6.

Handbook of Mathematical Functions, (1964). National Bureau of Standards.

Shenton, L. R. and Bowman, K. O. (1971). Continued fractions for the Psi-function and its derivatives, *SIAM Jour. Appl. Math.*, 20(4), 547-554.

Shenton, L. R. and Bowman, K. O. (2005). Second order continued fractions and Fibonacci numbers, *Far East Journal of Applied Mathematics*, 20(1), 17-31.

Stieltjes, T. J. (1905). *Correspondence D'Hermite et de Stieltjes*, Gauthier-Villars, Paris.

Stieltjes, T. J. (1918).*Oeuvres Complétes*, Tome 2, P. Noordhoff, Groningen.

Wall, H. S. (1948). *Continued Fractions*, Chelsea Publishing Co., Bronx, N.Y.