POLYGAMMA FUNCTIONS, THE ZETA FUNCTIONS, AND EULER NUMBERS

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Abstract

A relation is shown between polygamma functions, expressed as c.f.s, and the Riemann zeta function. Also the integral formulation of polygamma functions is shown to be related to Euler numbers.

Key words: Adams' computation, Euler's constant, geometric distribution cumulants.

1 Introduction

For the polygamma functions Shenton and Bowman, (1971) give the following results:

$$\psi_m(z) = (-1)^{m+1}\left(\frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}}\right) + \frac{(-1)^{m+1}}{z^m}C_m(z^2) \quad (\Re(z) > 0)$$

for $m = 0, 1, \cdots$; also for $m = 0$, $\frac{(-1)!}{z^m}$ is replaced by $-\ln(z)$. Here the polygamma function

$$\psi_m(z) = \frac{d^{m+1}}{dz^{m+1}} \ln\Gamma(z),$$

$\psi(z) = \psi_0(z)$ being the psi(digamma) function. Moreover there is the c.f.

$$C_m(z^2) = \frac{c_0^{(m)}}{z^2+}\ \frac{c_1^{(m)}}{1+}\ \frac{c_2^{(m)}}{z^2+}\ \frac{c_3^{(m)}}{1+}\ \cdots,$$

where $c_0^{(m)} = (m+1)!/12$, and $c_s^{(m)}$ is given to $s = 5$; further values could be found although there are complications (see Shenton and Bowman (1971, p.553)). The c.f.s are

$$\psi(z) = \ln z - \frac{1}{2z} - \left\{\frac{\frac{1}{12}}{z^2+}\ \frac{\frac{1}{10}}{1+}\ \frac{\frac{79}{210}}{z^2+}\ \frac{\frac{1205}{1659}}{1+}\ \frac{\frac{262445}{209429}}{z^2+}\ \frac{\frac{2643428417511}{1429053441530}}{1+}\ \cdots\right\}$$

$$\psi_1(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{z}\left\{\frac{1/6}{z^2+}\ \frac{1/5}{1+}\ \frac{18/35}{z^2+}\ \frac{20/21}{1+}\ \frac{50/33}{z^2+}\ \frac{315/143}{1+}\ \cdots\right\}$$

$$\psi_3(z) = \frac{2}{z^3} + \frac{3}{z^4} + \frac{1}{z^3}\left\{\frac{2}{z^2+}\ \frac{1/2}{1+}\ \frac{5/6}{z^2+}\ \frac{22/15}{1+}\ \frac{116/55}{z^2+}\ \frac{942/319}{1+}\ \cdots\right\}$$

$$\psi_4(z) = -\frac{6}{z^4} - \frac{12}{z^5} - \frac{1}{z^4}\left\{\frac{10}{z^2+}\ \frac{\frac{7}{10}}{1+}\ \frac{\frac{71}{70}}{z^2+}\ \frac{\frac{870}{497}}{1+}\ \frac{\frac{15092}{6177}}{z^2+}\ \frac{\frac{11140610}{3329403}}{1+}\ \cdots\right\}$$

$$\psi_5(z) = \frac{24}{z^5} + \frac{60}{z^6} + \frac{1}{z^5}\left\{\frac{60}{z^2+}\ \frac{\frac{14}{15}}{1+}\ \frac{\frac{127}{105}}{z^2+}\ \frac{\frac{3645}{1778}}{1+}\ \frac{\frac{21971635}{7838694}}{z^2+}\ \frac{\frac{145316386786}{38746664420}}{1+}\ \cdots\right\}$$

$$\psi_6(z) = -\frac{120}{z^6} - \frac{360}{z^7} - \frac{1}{z^6}\left\{\frac{420}{z^2+}\ \frac{\frac{6}{5}}{1+}\ \frac{\frac{149}{105}}{z^2+}\ \frac{\frac{14795}{6258}}{1+}\ \frac{\frac{2811525}{881782}}{z^2+}\ \frac{\frac{271790214451}{65240845725}}{1+}\ \cdots\right\}$$

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^2}\left\{\frac{1/3}{z^2+}\ \frac{2/3}{1+}\ \frac{6/5}{z^2+}\ \frac{9/5}{1+}\ \frac{18/7}{z^2+}\ \frac{24/7}{1+}\ \cdots\right\}$$

Now $\psi_2(z)$ is given by Stieltjes (1918, p385),

$$\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^2}\left\{\frac{1/2}{z^2+}\ \frac{p_1}{1+}\ \frac{q_1}{z^2+}\ \frac{p_2}{1+}\ \frac{q_2}{z^2}\ \cdots\right\}$$

where $p_s = \frac{s^2(s+1)}{4s+2}$, $q_s = \frac{s(s+1)^2}{4s+2}$, $s = 1, 2, \cdots$. As far as we know this is the only case for which all partial numerators are known. Indeed for the general case from Shenton and Bowman (1971, p553),

$$(-1)^{m+1}\psi_m(z) = \frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \frac{(m+1)!}{12z^m}\left\{\frac{1}{z^2+b_1^{(m)}-}\ \frac{a_1^{(m)}}{z^2+b_2^{(m)}-}\ \cdots\right\}$$

$$= \frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \frac{(m+1)!}{12z^m}\left\{\frac{1}{z^2+}\ \frac{c_1^{(m)}}{1+}\ \frac{c_2^{(m)}}{z^2+}\ \frac{c_3^{(m)}}{1+}\ \cdots\right\},$$

$m = 1, 2, \cdots$, where

$$a_1^{(m)} = \frac{(m+2)(m+3)\pi_2(m)}{5\cdot 7!}, \qquad a_2^{(m)} = \frac{(m+4)(m+5)\pi_6(m)}{11088\pi_2^2(m)};$$

$$b_1^{(m)} = \frac{(m+2)(m+3)}{60}, \qquad b_2(m) = \frac{m^4+60m^3+762m^2+3455m+5226}{30\pi_2(m)},$$

$$b_3^{(m)} = \frac{\pi_{10}(m)}{780\pi_2(m)\pi_6(m)}; \qquad c_1^{(m)} = b_1^{(m)},$$

$$c_2^{(m)} = \frac{\pi_2(m)}{420}, \qquad c_3^{(m)} = \frac{(m+4)(m+5)\pi_2^*(m)}{84\pi_2(m)},$$

$$c_4^{(m)} = \frac{\pi_6(m)}{132\pi_2(m)\pi_2^*(m)}, \qquad c_5^{(m)} = b_3^{(m)} - c_4^{(m)};$$

and

$$\pi_2(m) = 3m^2+55m+158, \qquad \pi_2^*(m) = m^2+93m+482,$$

$$\pi_6(m) = m^6+723m^5+22659m^4+339313m^3+2581460m^2+934168m+12597360,$$

$$\pi_{10}(m) = 10m^{10}+40518m^9+270571m^8+89788404*m^7+1794433554m^6$$
$$+22947479418m^5+190840554986m^4+1020244044156m^3$$
$$+3360583420456m^2+6173053021104m+4817280991680.$$

The fifth partial numerator involves $\pi_6(m)$, not an attractive prospect. Another examples arises from the well known expression,

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln(z) - z + \frac{1}{2}\ln(2\pi) + J(z), \quad (\Re(z) > 0)$$

where

$$J(z) = \frac{1}{\pi}\int_0^\infty \ln\left(\frac{1}{1-e^{-2\pi t}}\right)\frac{zdt}{(z^2+t^2)} = \frac{a_1}{z+}\,\frac{a_2}{z+}\,\frac{a_3}{z+}\,\frac{a_4}{z+}\,\frac{a_5}{z+}\,\cdots,$$

and

$$a_1 = 1/12, \quad a_2 = 1/30, \quad a_3 = 53/210,$$

$$a_4 = 195/371, \quad a_5 = 22999/22737,$$

the a_i being positive. This was given by Stieltjes (1905, p351, letter of 1889). Wall (1948) gives a_6 and a_7. These partial numerators do not seem to subscribe to simple closed forms. But as already noted, by contrast, for $\psi_2(z)$ all partial numerators are known in simple closed forms. Perhaps J.B.S. Haldane would have remarked "Mathematics is not only curious but more curious than we can think".

In this paper we use the recurrence $\Gamma(x+1) = x\Gamma(x)$ and its derivatives to establish relations between polygamma functions and the Riemann zeta function.

2 Polygamma functions and the Zeta function

We have

$$\psi(z+n) - \psi(z) = \sum_{s=0}^{n-1}\frac{1}{z+s} \quad (\Re(z) > 0, n = 1, 2, \cdots)$$

which is derived from

$$\psi(z+1) - \psi(z) = \frac{1}{z}.$$

Differentiating m times we have

$$- \psi_m(z) = (-1)^m m! \sum_{s=0}^{n-1} \frac{1}{(z+s)^{m+1}} - \psi_m(z+n)$$
$$= (-1)^m m! \sum_{s=0}^{n-1} \frac{1}{(z+s)^{m+1}} + (-1)^m \left\{ \frac{(m-1)!}{(z+n)^{m+1}} + \frac{m!}{2(z+n)^{m+1}} + \frac{C_m\{(z+n)^2\}}{(z+n)^m} \right\}.$$

$$(m = 0, 1, \cdots; n = 1, 2, \cdots; \Re(z) > 0)$$

If $m = 0$, $z = 1$, then

$$-\psi(1) = (1 + \frac{1}{2} + \cdots + \frac{1}{n}) + \frac{1}{2(n+1)} - \ln(n+1) + C_0(n+1)^2. \quad (n \to \infty)$$

Now $C_0(n+1)^2 = \frac{\frac{1}{12}}{N^2+} \frac{\frac{1}{10}}{1+} \frac{\frac{79}{210}}{N^2+} \frac{\frac{1205}{1659}}{1+} \cdots$, $N = n + 1$
from §1. This we set up the tabulation of approximants to γ.

Table 1. Bounds for Euler's Constant with $n = 1$ and successive terms of the c.f.

s	Using s partial numerators	|Error|
1	0.5776861528	0.0004704879
2	0.5771780227	0.0000376422
3	0.5772207272	0.0000050623
4	0.5772146373	0.0000010276
5	0.5772159268	0.0000002619
6	0.5772155823	0.0000000826

The first approximant is $5/4 - \ln 2 + 1/48$, second is $5/4 - \ln 2 + 5/246$.

Clearly the c.f. approximants are quite sharp, the odd convergents providing a set of decreasing bounds, the even a set of increasing bounds. Note that in the 1971 paper we have provided that all partial numerators are positive. The general expression for $\psi_m(1)$ may be used to set up similar bounds for (i) $\psi_1(1)$ which is related to the zeta function $\zeta(2) = \pi^2/6$, (ii) $\psi_3(1)$ which is related to $\pi^4/90$, and (iii) $\psi_{2s-1}(1)$ which is related to $\zeta(2s) = \frac{(2\pi)^{2s}}{2(2s)!}|B_{2s}|$.

J.C. Adams (1878) computed γ, Euler's constant to 264 decimal places. His result being

E=	.57721	56649	01532	86060	65120	90082	40243	10421	59335	93992
	35988	05767	23488	48677	26777	66467	09369	47063	29174	67495
	14631	44724	98070	82480	96050	40144	86542	83622	41739	97644
	92353	62535	00333	74293	73377	37673	94279	25952	58247	09491
	60087	35203	94816	56708	53233	15177	66115	28621	19950	15079
	84793	74508	5697							(E=γ)

Adams had to compute the Bernoulli number B_{62}; he gives an interesting account of Euler's use of divergent series. His value for γ is correct up to 261 digits according to the value computed on Maple.

In our 1971 paper on derivatives of the psi function we give the Stieltjes transforms

$$\begin{aligned}
\psi(z) &= \ln z - \frac{1}{2z} - \int_0^\infty \frac{dx}{(y-1)(z^2+x)}, \\
\psi_1(z) &= \frac{1}{z} + \frac{1}{2z^2} + \left(\frac{2\pi}{z}\right) \int_0^\infty \frac{y\sqrt{x}dx}{(y-1)^2(z^2+x)}, \\
\psi_2(z) &= -\frac{1}{z^2} - \frac{1}{z^3} - \left(\frac{2\pi}{z}\right)^2 \int_0^\infty \frac{(y+y^2)dx}{(y-1)^3(z^2+x)}, \\
\psi_3(z) &= \frac{2}{z^3} + \frac{3}{z^4} + \left(\frac{2\pi}{z}\right)^3 \int_0^\infty \frac{(y+4y^2+y^3)x^{3/2}dx}{(y-1)^4(z^2+x)}. \quad (\Re(z)>0)
\end{aligned} \tag{1}$$

The general case will be

$$\psi_m(z) = (-1)^{m+1}\left\{\frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \left(\frac{2\pi}{z}\right)^m \int_0^\infty \frac{x^{m/2}\Theta_m(y)dx}{(z^2+x)(y-1)^{m+1}}\right\}, \tag{2}$$

where $y = \exp(2\pi\sqrt{x})$, and $\Theta_m(y)$ is perhaps a polynomial in y of degree m.

Now there is the asymptotic expansion

$$\psi_m(z) \sim (-1)^{m+1}\left\{\frac{(m-1)!}{z^m} + \frac{m!}{2z^{m+1}} + \sum_{k=1}^{\infty} \frac{(2k+n-1)!}{(2k)!z^{2k+m}} B_{2k}\right\}$$

for $z \to \infty$ in $|\arg z| < \pi$. Therefore the coefficient of $1/z^{2k+m}$ is

$$(-1)^{m+1}\frac{(2k+m-1)!B_{2k}}{(2k)!}.$$

This implies that

$$(2\pi)^m \int_0^\infty \frac{x^{\frac{m}{2}+k-1}\Theta_m(y)dx}{(y-1)^{m+1}} = \frac{(2k+m-1)!|B_{2k}|}{(2k)!}, \quad (m=1,2,\cdots; k=1,2,\cdots) \tag{3}$$

where B_k is a Bernoulli number.

3 An expression for $\Theta_m(y)$ involving Euler numbers

Now in connection with a related problem, Shenton and Bowman (2001) used the formula

$$\Theta_m(t) = (1-e^{-t})^{m+1}\left(\frac{d}{dt}\right)^m \frac{1}{1-e^{-t}}. \quad (t>0, m=0,1,\cdots)$$

For examples

$$\Theta_0(t) = 1, \qquad \Theta_1(t) = e^{-t},$$
$$\Theta_2(t) = e^{-t} + e^{-2t}, \qquad \Theta_3(t) = e^{-t} + 4e^{-2t} + e^{-3t}.$$

Now with $y = e^{-t}$ in the integrals of (3), there is agreement corresponding to the forms in (1). Hence

$$\frac{\Theta_m(t)}{(1-e^{-t})^{m-1}} = \left(\frac{d}{dt}\right)^m \frac{1}{1-e^{-t}} = 1^m e^{-t} + 2^m e^{-2t} + 3^m e^{-3t} + \cdots.$$

Returning to (3), the left side becomes, using $1/y = e^{-t}$,

$$= \frac{2}{(2\pi)^{2k}}(2k+m-1)!\left\{1 + \frac{2^m}{2^{m+2k}} + \frac{3^m}{3^{m+2k}} + \frac{4^m}{4^{m+2k}} + \cdots\right\},$$
$$= \frac{2}{(2\pi)^{2k}}(2k+m-1)!\zeta(2k),$$
$$= \frac{2}{(2\pi)^{2k}}(2k+m-1)!\frac{(2\pi)^{2k}}{2(2k)!}|B_{2k}|,$$
$$= \frac{(2k+m-1)!|B_{2k}|}{(2k)!},$$

as it should.

The $\Theta_m(e^{-t})$ polynomials turn up in expressions for the cumulants of a geometric random variable.

4 Conclusion

At least formally when $\psi_m(z)$ is expressed in terms of a Stieltjes integral (expression (1)), we have shown that the integrand has a polynomial component whose coefficients are Euler numbers, such as

$$
\begin{aligned}
k_1^{(m)} &= 1^{(m)}, \\
k_2^{(m)} &= 2^m - (m+1)1^m, \\
k_3^{(m)} &= 3^m - \binom{m+1}{1} 2^m + \binom{m+1}{2} 1^m, \\
k_s^{(m)} &= \sum_{u=0}^{s-1} (-1)^u \binom{m+1}{u} (s-u)^m, \quad (m = 1, 2, \cdots; s = 1, 2, \cdots, m),
\end{aligned}
$$

Riordan (1958).

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