Multiscale discretizations for multiphase porous media flow coupled with geometrics

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1. Poroelastic Model Problem

Biot Model for poroelasticity describe fluid flow in a deformable porous media, allowing the computation of compaction, subsidence, uplift, and damage.

Geometrics:

$\nabla \cdot \boldsymbol{\sigma} = \mathbf{f}$ $\quad$ $\mathbf{u}$ = displacement

$\boldsymbol{\sigma} = \lambda (\nabla \cdot \mathbf{u}) \mathbf{I} + 2\mu \boldsymbol{\epsilon} - \alpha p \mathbf{I}$ $\quad$ $\boldsymbol{\epsilon}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ = strain

Data: body force $\mathbf{f}$, Lamé parameters λ and μ, Biot constant α.

Single phase flow:

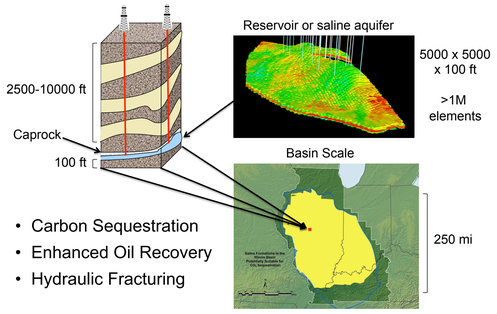
$\frac{\partial}{\partial t}(c_o p + \alpha \nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{z} = q$ $\quad$ p = pore pressure

$\mathbf{z} = -\mathbf{K}(\nabla p - \rho \mathbf{g})$ $\quad$ $\mathbf{z}$ = fluid velocity.

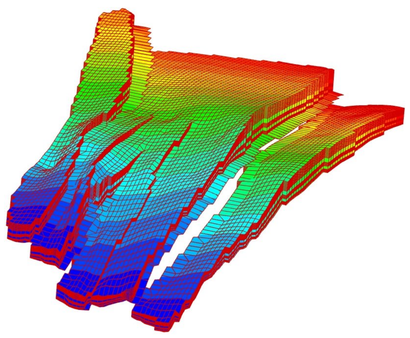
Data: volumetric source term q, permeability $\mathbf{K}$, gravity $\mathbf{g}$, density $\rho = \rho(p)$, storage coefficient c_0.

1.1 Applications

Very large scale applications:



- Carbon Sequestration
- Enhanced Oil Recovery
- Hydraulic Fracturing



A distorted hexahedral grid for the Frio CO_2 Pilot Test.

1.2 Modeling Goals

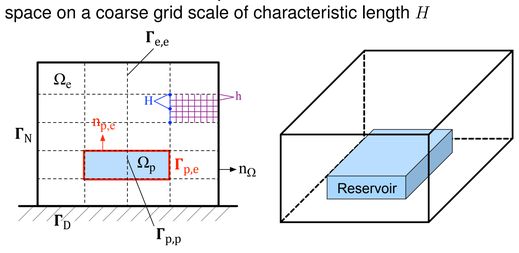
- Locally conservative discretization
- Capable of handing rough geometry
- Multiscale and multi-physics
- Parallel framework using domain decomposition
- Efficient solvers

1.3 Multiscale Mortar Modeling Methodology

Ω_p = poroelastic region with both mechanics and fluid flow

Ω_e = elastic region with mechanics and no-flow ($\alpha = 0$)

- Ω_p and Ω_e are divided into non-overlapping subdomains with nonmatching grids of characteristic length h
- Interface conditions are imposed via a mortar finite element space on a coarse grid scale of characteristic length H



Interface conditions:

Continuity of medium:	$[\mathbf{u}] = \mathbf{0}$	on $\Gamma_{e,e}, \Gamma_{p,p}, \Gamma_{p,e}$
Continuity of normal stress:	$[\sigma(\mathbf{u})]\,\mathbf{n} = 0$	on $\Gamma_{e,e}, \Gamma_{p,p}, \Gamma_{p,e}$
No-flow outside pay-zone:	$\mathbf{z} \cdot \mathbf{n} = 0$	on $\Gamma_{p,e}$
Continuity of pressure:	$[p] = 0$	on $\Gamma_{p,p}$
Continuity of normal flux:	$[\mathbf{z} \cdot \mathbf{n}] = 0$	on $\Gamma_{p,p}$

- The algebraic system is reduced to a coarse scale mortar interface problem that is solved efficiently using a multiscale flux basis

2. Multipoint Flux Mixed Finite Element (MFMFE) Method

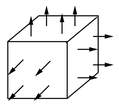
- Mixed finite element variational formulation
- Reduces to a cell centered method on general hexahedral meshes
- First order accurate; second order fluxes on smooth grids

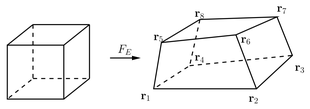
Piola transformation maps MFE space on a reference element to the hexahedral mesh, preserving normal velocity components.

Enhanced BDDF$_1$ space

$\mathbf{V}_h$: 4 DOF per face (bilinear)

W_h: const





$$DF_{ij} = \frac{\partial F_i}{\partial \hat{x}_j}, \quad J = |\det(DF)|, \quad \mathbf{V}_h(E) = \frac{1}{J_E} DF_E \hat{\mathbf{V}}(\hat{E})$$

Find $\mathbf{z}_h \in \mathbf{V}_h$ and $u_h \in W_h$ such that

$$(K^{-1}\mathbf{z}_h, \mathbf{v})_Q - (u_h, \nabla \cdot \mathbf{v}) = 0, \qquad \forall \mathbf{v} \in \mathbf{V}_h,$$
$$(\nabla \cdot \mathbf{z}_h, w) = (f, w), \qquad \forall w \in W_h$$

Quadrature rule $(\cdot,\cdot)_Q$ allows to eliminate the flux in terms of the pressure degrees of freedom.

$$(K^{-1}\mathbf{q}, \mathbf{v})_E = (\mathcal{M}_E \hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}}, \qquad \mathcal{M}_E = \frac{1}{J_E} DF_E^T K^{-1} DF_E$$

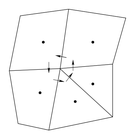
Symmetric numerical quadrature (smooth grids):

$$(K^{-1}\mathbf{q}, \mathbf{v})_{Q,E} := \text{Trap}\,(\mathcal{M}_E \hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}} = \frac{1}{8} \sum_{i=1}^{8} \mathcal{M}_E(\hat{\mathbf{r}}_i)\hat{\mathbf{q}}(\hat{\mathbf{r}}_i) \cdot \hat{\mathbf{v}}(\hat{\mathbf{r}}_i)$$

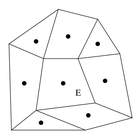
Non-symmetric quadrature (rough grids):

$$\widetilde{\mathcal{M}}_E = \frac{1}{J_E} DF_E(\hat{\mathbf{r}}_c) \overline{K}^{-1} DF_E, \quad \hat{\mathbf{r}}_c: \text{ center of } \hat{E}, \ \overline{K}: \text{ average of } K.$$

$$(K^{-1}\mathbf{q}, \mathbf{v})_{Q,E} \equiv \text{Trap}(\widetilde{\mathcal{M}}_E \hat{\mathbf{q}}, \hat{\mathbf{v}})_{\hat{E}}$$



Local velocity elimination



Cell-centered pressure stencil

2.1 Theoretical Results

Theorem

There exists a constant C independent of h such that

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(H^1)} + \|\mathbf{z} - \mathbf{z}_h\|_{L^2(L^2)} + \|p - p_h\|_{L^\infty(L^2)} \le Ch\left(\|\mathbf{u}\|_{H^1(H^2)} + \|p\|_{H^1(H^1)} + \|\mathbf{z}\|_{L^2(H^1)}\right).$$

2.2 Flow and geomechanics in 2D

Analytical Solutions:

$$\mathbf{u} = \begin{pmatrix} \sin(\pi t)x(1-x)y(1-y) \\ \sin(\pi t)x(1-x)y(1-y) \end{pmatrix}$$

$$p = t\sin(2\pi x)\sin(2\pi y)$$

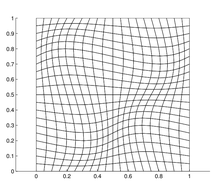
Parameters:

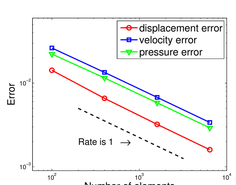
- $\alpha = 1$
- $\nu = 0.2 \quad E = 1$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \mu = \frac{E}{2(1+\nu)}$$

- $c_o = 0.1 \quad \mathbf{K} = 1 \quad \mu_f = 1 \quad \mathbf{g} = 0$

Final Time: $T = 0.1$, $\quad$ Time Step Size: $\Delta t = 10^{-3}$.



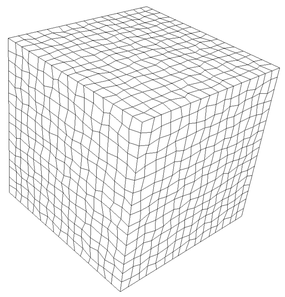


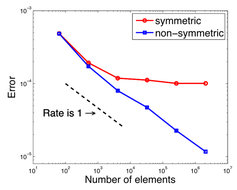
2.3 Single phase flow in 3D

3D distorted hexahedra with non-planar faces

$$p(x,y,z) = x^2(x-1)^2 y^2 (y-1)^2 z^2 (z-1)^2$$

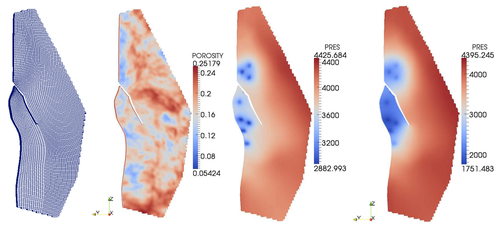
$$K = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$





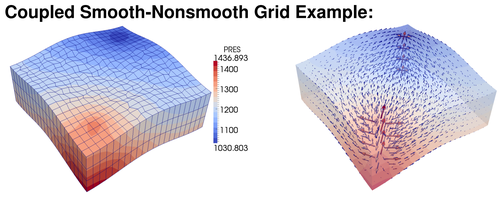
2.4 Single phase compressible flow

Brugge Example:

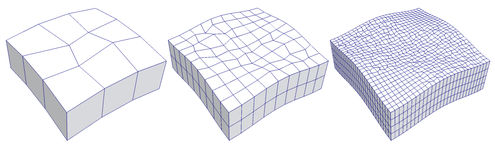


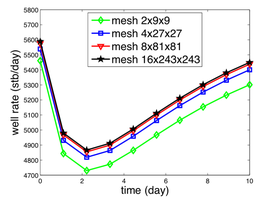
From left to right: computational mesh, porosity field, pressure at day 10, pressure at day 100.

Coupled Smooth-Nonsmooth Grid Example:



Pressure $\qquad$ Velocity field





The total production rate is shown for a series of mesh refinements.

2.5 Multiscale Mortar MFMFE Method

Find $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, and $\lambda_H \in M_H$ such that

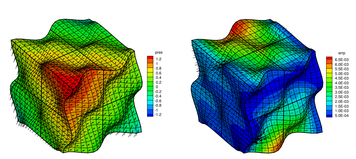
$$(K^{-1}\mathbf{u}_h, \mathbf{v})_{Q,\Omega_i} - (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} = -\langle \lambda_H, \Pi^{RT}\mathbf{v} \cdot \mathbf{n}_i \rangle_{\Gamma_i}, \quad \forall \mathbf{v} \in \mathbf{V}_{h,i},$$
$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad \forall w \in W_{h,i},$$
$$\sum_{i=1}^{n} \langle \Pi^{RT}\mathbf{u}_h \cdot \mathbf{n}_i, \mu \rangle_{\Gamma_i} = 0, \quad \forall \mu \in M_H.$$

Theorem:

$$\|\mathbf{u} - \mathbf{u}_h\| + \|p - p_h\| \le C(H^{m+1/2} + h), \quad \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \le Ch$$
$$\|\mathbf{u} - \Pi^{RT}\mathbf{u}_h\| \le C(H^{m+1/2} + hH^{1/2}), \quad \|p - p_h\| \le C(H^{m+3/2} + hH)$$

Example:

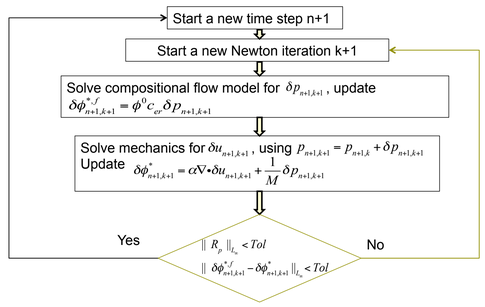
$$p(x,y,z) = x + y + z - 1.5,$$
$$K = \begin{pmatrix} x^2+y^2+1 & 0 & 0 \\ 0 & z^2+1 & \sin(xy) \\ 0 & \sin(xy) & x^2y^2+1 \end{pmatrix}$$
$$x = \hat{x} + 0.03\cos(3\pi\hat{x})\cos(3\pi\hat{y})\cos(3\pi\hat{z}),$$
$$y = \hat{y} - 0.04\cos(3\pi\hat{x})\cos(3\pi\hat{y})\cos(3\pi\hat{z}),$$
$$z = \hat{z} + 0.05\cos(3\pi\hat{x})\cos(3\pi\hat{y})\cos(3\pi\hat{z}).$$

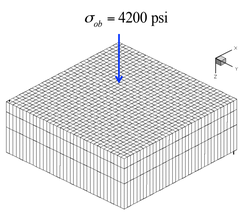


3. Parallel Simulation of Compositional Flow Coupled with Geomechanics

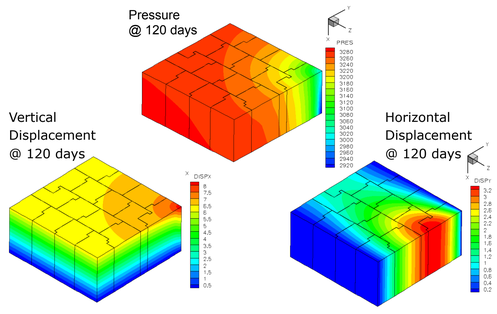
$\frac{\partial(\phi N_i)}{\partial t} = -\nabla \cdot (\sum_\alpha \vec{J}_i^\alpha) + q_i$	Mass balance equation
$\vec{J}_i^\alpha = \rho_\alpha \xi_i^\alpha \vec{u}_\alpha - \phi \rho_\alpha S_\alpha \underline{\underline{D}}_i^\alpha \nabla \xi_i^\alpha$	Flux of component i in phase α
$\vec{u}_\alpha = -\frac{k_{r\alpha}}{\mu_\alpha} \underline{\underline{K}}(\nabla p_\alpha - \rho_\alpha \vec{g})$	Velocity of phase α
$p_\alpha = p + p_{c\alpha}(S_\alpha)$	Fluid Pressure for phase α
$\phi = \phi_0\{1 + c_r(p - p_0)\}$	Porosity
$\xi_i^\alpha = \xi_i^\alpha(p, T, N_i)$	Mole fraction of component i in phase α
$\mu_\alpha = \mu_\alpha(p, T, \xi_i^\alpha)$	Viscosity of phase α
$S_\alpha = S_\alpha(p, N_i, \xi_i^\alpha, T)$	Saturation of phase α
$\rho_\alpha = \rho_\alpha(p, T, \xi_i^\alpha)$	Molar density of phase α
$k_{r\alpha} = k_{r\alpha}(S_\alpha)$	Relative permeability for phase α

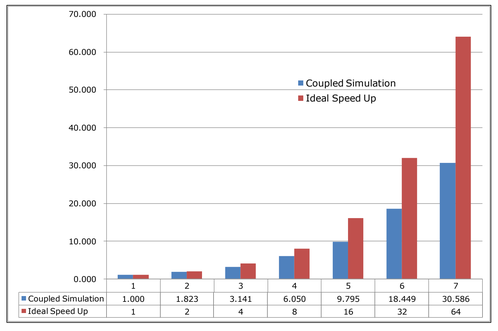
3.1 Numerical Example





Dimension	3500 X 3500 X 100 (ft)
Grids	35 X 35 X 3
Components	C1, C3, C6, C10, C15, C20
Initial pressure (psi)	3984.3, 3990.3, 4000.0
Porosity	0.3
Oil production rate (STB/D)	12000
Initial water saturation	0.2
X, Y-perm (md)	500, 50, 200
Z-perm (md)	50, 50, 25
Young's modulus (psi)	1.E5
Poisson's ratio	0.3





4. Future Work

- The compositional flow model is currently being rewritten to use the MFMFE method.
- Various methods for coupling linear elastic domains with mortars are currently being explored.

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