PDF Methods for Uncertainty Quantification in Hyperbolic Conservation Laws

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Abstract

We developed a probabilistic method to quantify uncertainty associated with hyperbolic conservation laws. The approach relies on the concept of fine-grained probability density function (PDF) or cumulative density function (CDF) and derives a deterministic equation for the PDF/CDF of the system state. It is computationally efficient and enables one to obtain full statistical description of system states, which is necessary for risk assessment. We demonstrate three such examples, namely, advection-reaction equation, kinematic wave equation and advective-diffusive transport, respectively.

Hyperbolic conservation laws

$$\frac{\partial c}{\partial t} + \nabla \cdot \mathbf{f}(c;\mathbf{x},t) = r(c;\mathbf{x},t)$$

PDF Methods for Parametric Uncertainty

- Deterministic PDE of system states PDFs
- No linearization of SPDEs
- Complete statistics of system states
- Computationally efficient and accurate

Raw Distribution:

$$\Pi(c, C; x, t) = \delta[C - c(x,t)]$$

Probability Density Function (PDF):

$$\overline{\Pi}(c, C; x, t) = p_c(C; x, t)$$

Advective-Reactive Transport

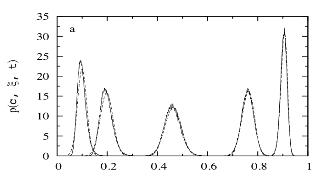
$$\frac{\partial c}{\partial t} = -\nabla \cdot (\mathbf{u}c) + \alpha f_\alpha(c), \quad f_\alpha = -k(c^\alpha - C_{eq}^\alpha)$$

Sources of Uncertainty:

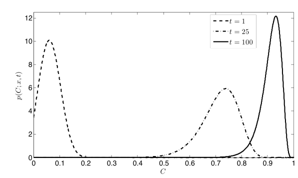
Velocity field *u(x,t,ω)* & Reaction rate *k(x,ω)*

PDF Equation:

$$\frac{\partial p_c}{\partial t} = -\frac{\partial \tilde{u}_i p_c}{\partial \tilde{x}_i} + \frac{\partial}{\partial \tilde{x}_k}\left[\tilde{D}_{ij}\frac{\partial p_c}{\partial \tilde{x}_i}\right], \quad \begin{cases} \tilde{\mathbf{x}} = (x_1, x_2, x_3, C)^T \\ \tilde{\mathbf{u}} = (u_1, u_2, u_3, f_\alpha)^T \end{cases}$$



Temporal evolution of the concentration PDF with deterministic velocity at increasing times (from left to right) using random walk simulations (solid lines) and Gaussian approximation (dashed lines). [1]



Temporal evolution of the concentration PDF for the linear reaction law (*α=1*), spatially uncorrelated reaction rate constant *κ(x)*.[2]

Kinematic Wave (Saint-Venant) Equation

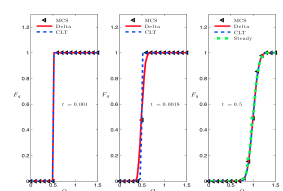
$$\frac{\partial c}{\partial t} + \frac{\partial q}{\partial x} = r(x,t), \qquad q = \alpha c^{1/\beta}$$

Sources of Uncertainty:

- Surface slope and resistance *α(x,t)*
- Measurement of turbulence *β*
- Sources *r(x,t)*
- Initial and boundary conditions

Effective CDF Equation:

$$\frac{\partial F_q}{\partial t} + \mathbf{v}_{\text{eff}} \cdot \nabla_{\mathbf{x}} F_q = \nabla_{\mathbf{x}} \cdot (\mathbf{D}\nabla_{\mathbf{x}} F_q)$$



Flow rate CDF computed with MCS, the white noise approximation (Delta), and the CLT-based approximation (CLT). [3]

Advective-Diffusive Transport

$$\frac{\partial c}{\partial t} = D\nabla^2 c - \nabla \cdot (\mathbf{u}c)$$

Sources of Uncertainty:

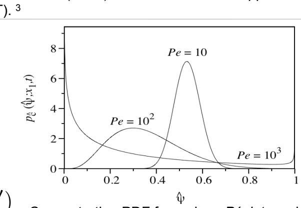
Velocity field *u(x,t,ω)*

Map $p_c(\psi, x, t)$ onto two observables:

- Center of mass of plume: $\mathbf{m}(t) \sim \int c(\mathbf{x},t)\mathbf{x}\,\mathrm{d}x_1 \cdots \mathrm{d}x_d$
- Width of plume: $k_{ij}(t) \sim \int c(\mathbf{x},t)x_i x_j\,\mathrm{d}x_1 \cdots \mathrm{d}x_d$

Particle trajectories x(t) follows a Lanegvin Equation:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{u}[\mathbf{x}(t), t] + \boldsymbol{\xi}(t), \; \langle \xi_i(t)\xi_j(t')\rangle_\xi = 2D_{ij}\delta(t-t')$$



Concentration PDF for various *Péclet* numbers.[4]

Conclusions

- While standard techniques for uncertainty quantification typically yield only system state's mean and variance, the proposed approach leads to its full probabilistic description
- The shape of the PDF changes with time, varying between the known initial and steady-state distributions. This makes reliance on assumed PDFs problematic.
- PDF methods provide a computationally efficient means for uncertainty quantification.

References:

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Acknowledgment

This research was supported by Department of Energy ASCR, Applied Mathematics.

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