Positivity-preserving high-order Runge-Kutta discontinuous Galerkin schemes

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Introduction

For the scalar conservation laws

$$u_t + \nabla \cdot \mathbf{F}(u) = 0, \qquad u(\mathbf{x}, 0) = u_0(\mathbf{x}). \tag{1}$$

An important property of the entropy solution (which may be discontinuous) is that it satisfies a strict maximum principle: If

$$M = \max_{\mathbf{x}} u_0(\mathbf{x}), \quad m = \min_{\mathbf{x}} u_0(\mathbf{x}), \tag{2}$$

then $u(\mathbf{x}, t) \in [m, M]$ for any $\mathbf{x}$ and t.

First order monotone schemes can maintain the maximum principle. However, no higher order linear schemes can satisfy the maximum principle (Godunov Theorem). Therefore, nonlinear schemes have been designed. These include roughly two classes of schemes:

- TVD schemes. Most TVD (total variation diminishing) schemes also satisfy strict maximum principle, even in multi-dimensions. TVD schemes can be designed for any formal order of accuracy for solutions in smooth, monotone regions. However, all TVD schemes will degenerate to first order accuracy at smooth extrema.

- TVB schemes, ENO schemes, WENO schemes. These schemes do not satisfy strict maximum principles, although they can be designed to be arbitrarily high order accurate for smooth solutions.

The flowchart for designing a high order discontinuous Galerkin scheme which obeys a strict maximum principle is as follows:

1. Start with $u^n(x)$ which is high order accurate

$$|u(x, t^n) - u^n(x)| \le C\Delta x^p$$

   and satisfy

$$m \le u^n(x) \le M, \qquad \forall x$$

   therefore of course we also have

$$m \le \bar{u}_j^n \le M, \qquad \forall j.$$

2. Evolve for one time step to get $u^{n+1,prel}(x)$ with its cell averages satisfying

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j. \tag{3}$$

3. Given (3) above, limit $u^{n+1,prel}(x)$ to obtain $u^{n+1}(x)$ (without changing the cell averages) which

   - satisfies the maximum principle

$$m \leq u^{n+1}(x) \leq M, \qquad \forall x;$$

   - is high order accurate

$$|u(x, t^{n+1}) - u^{n+1}(x)| \leq C\Delta x^p.$$

Three major difficulties

1. The first difficulty is how to evolve in time for one time step to guarantee

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j. \tag{4}$$

This is very difficult to achieve. Previous works use one of the following two approaches:

- Use exact time evolution. This can guarantee

$$m \le \bar{u}_j^{n+1} \le M, \qquad \forall j.$$

However, it can only be implemented with reasonable cost for linear PDEs, or for nonlinear PDEs in one dimension. This approach was used in, e.g., Jiang and Tadmor, SISC 1998; Liu and Osher, SINUM 1996; Sanders, Math Comp 1988; Qiu and Shu, SINUM 2008; Zhang and Shu, SINUM 2010; to obtain TVD schemes or maximum-principle-preserving schemes for linear and nonlinear PDEs in one dimension or for linear PDEs in multi-dimensions, for second or third order accurate schemes.

- Use simple time evolution such as SSP Runge-Kutta or multi-step methods. However, additional limiting will be needed on $u^n(x)$ which will destroy accuracy near smooth extrema.

We have figured out a way to obtain

$$m \le \bar{u}_j^{n+1} \le M, \qquad \forall j$$

with simple Euler forward or SSP Runge-Kutta or multi-step methods without losing accuracy on the limited $u^n(x)$.

2. The second difficulty is: given

$$m \leq \bar{u}_j^{n+1} \leq M, \qquad \forall j$$

how to obtain accurate reconstruction $u^{n+1}(x)$ which satisfy

$$m \leq u^{n+1}(x) \leq M, \qquad \forall x.$$

Previous work was mainly for relatively lower order schemes (second or third order accurate), and would typically require an evaluation of the extrema of $u^{n+1}(x)$, which, for a piecewise polynomial of higher degree and in multi-dimensions, is quite costly.

We have figured out a way to obtain such reconstruction with a very simple limiter, which only requires the evaluation of $u^{n+1}(x)$ at certain pre-determined quadrature points and does not destroy accuracy.

3. The third difficulty is how to generalize the algorithm and result to 2D (or higher dimensions). Algorithms which would require an evaluation of the extrema of the reconstructed polynomials $u^{n+1}(x, y)$ would not be easy to generalize at all.

   Our algorithm easily generalizes to 2D or higher dimensions, with strict maximum-principle-satisfying property and provable high order accuracy.

High order schemes finite volume and DG schemes

Maximum-principle-satisfying DG and finite volume WENO schemes for scalar conservation laws and passive convection in an incompressible velocity field, and positivity-preserving (for density and pressure) DG and finite volume WENO schemes for compressible Euler equations (Zhang and Shu, SINUM 2010; JCP 2010a; JCP 2010b; Zhang, Xia and Shu, JSC to appear; Zhang and Shu, JCP 2011), gaseous detonations (Wang, Zhang, Shu and Ning, JCP to appear), and shallow water equations with mixed wet/dry regions (Xing, Zhang and Shu, Advances in Water Resources 2010; Xing and Shu, Advances in Water Resources 2011).

We have a scheme which, for one dimensional scalar conservation laws, satisfies a strict maximum principle and is uniformly high order accurate.

The technique has been generalized to the following situations maintaining uniformly high order accuracy:

- 2D scalar conservation laws on rectangular or triangular meshes with strict maximum principle (Zhang and Shu, JCP 2010a; Zhang, Xia and Shu, JSC to appear).

- 2D incompressible equations in the vorticity-streamfunction formulation (with strict maximum principle for the vorticity), and 2D passive convections in a divergence-free velocity field with strict maximum principle (Zhang and Shu, JCP 2010a; Zhang, Xia and Shu, JSC to appear).

- One and multi-dimensional compressible Euler equations maintaining positivity of density and pressure (Zhang and Shu, JCP 2010b, 2011; Zhang, Xia and Shu, JSC to appear).
- One and two-dimensional shallow water equations maintaining non-negativity of water height and well-balancedness for problems with dry areas (Xing, Zhang and Shu, Advances in Water Resources 2010; Xing and Shu, Advances in Water Resources 2011).

- One and multi-dimensional compressible Euler equations with gaseous detonations maintaining positivity of density, pressure and reactant mass fraction, with a new and simplified implementation of the pressure limiter. DG computations are stable without using the TVB limiter (Wang, Zhang, Shu and Ning, JCP to appear).

- Positivity-preserving for PDEs involving global integral terms including a hierarchical size-structured population model (Zhang, Zhang and Shu, JCAM to appear) and Vlasov-Boltzmann transport equations (Cheng, Gamba and Proft, Math Comp 2011).

- Positivity-preserving semi-Lagrangian schemes (Qiu and Shu, JCP to appear; Rossmanith and Seal, JCP to appear).

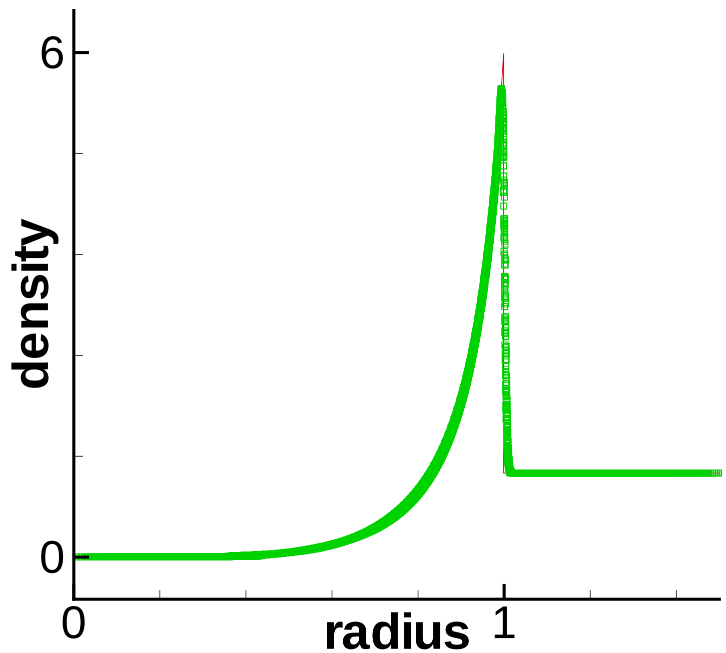
Numerical results

Example 1. Accuracy check. For the incompressible Euler equation in the vorticity-streamfunction formulation. We clearly observe the designed order of accuracy for this solution.

Table 1: Incompressible Euler equations. P^2 for vorticity, $t = 0.5$.

N×N	L^1 error	order	L^∞ error	order
16×16	5.12E-4	–	1.40E-3	–
32×32	3.75E-5	3.77	1.99E-4	2.81
64×64	3.16E-6	3.57	2.74E-5	2.86
128×128	2.76E-7	3.51	3.56E-6	2.94

Example 2. The Sedov point-blast wave in two dimensions. The computational domain is a square. For the initial condition, the density is 1, velocity is zero, total energy is 10^{-12} everywhere except that the energy in the lower left corner cell is the constant $\frac{0.244816}{\Delta x \Delta y}$. $\gamma = 1.4$. See Figure 1. The computational result is comparable to those in the literature, e.g. those computed by Lagrangian methods.



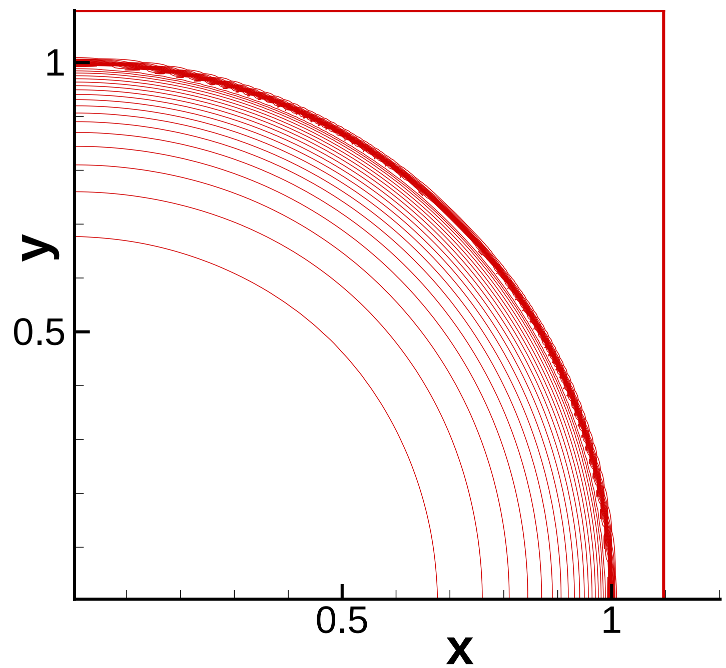


Figure 1: 2D Sedov blast, plot of density. $T = 1$. $N = 160$. $\Delta x = \Delta y = \frac{1.1}{N}$. TVB limiter parameters $(M_1, M_2, M_3, M_4) = (8000, 16000, 16000, 8000)$.

Example 3. Shock diffraction problem. Shock passing a backward facing corner. It is easy to get negative density and/or pressure below and to the right of the corner. The setup is the following: the computational domain is the union of $[0,1] \times [6,11]$ and $[1,13] \times [0,11]$; the initial condition is a pure right-moving shock of $Mach = 5.09$, initially located at $x = 0.5$ and $6 \leq y \leq 11$, moving into undisturbed air ahead of the shock with a density of 1.4 and pressure of 1. $\gamma = 1.4$ and the TVB limiter parameters $M_i = 100$ for $i = 1, 2, 3, 4$. The density and pressure at $t = 2.3$ are presented.

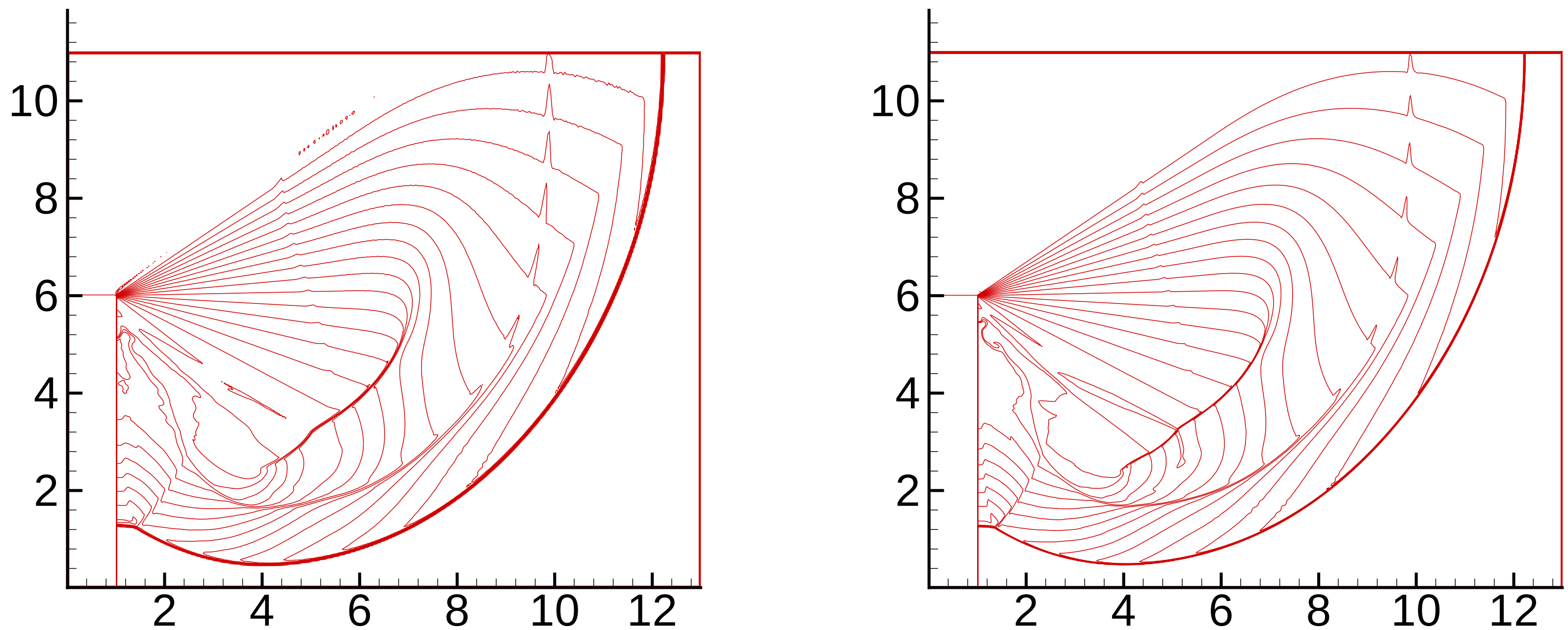


Figure 2: Shock diffraction problem. Density: 20 equally spaced contour lines from $\rho = 0.066227$ to $\rho = 7.0668$. Left: $\Delta x = \Delta y = 1/32$; Right: $\Delta x = \Delta y = 1/64$.

Concluding remarks

- We have obtained, for the first time, high order schemes for multi-dimensional nonlinear scalar conservation laws and passive convections in incompressible velocity fields that satisfy strict maximum principle, using an easy procedure involving only slight change from standard finite volume and DG schemes with SSP time discretizations.

- This technique has been generalized to 2D triangles, and to positivity preserving schemes for compressible Euler equations and shallow water equations.