Rare-event Splitting for Efficient Simulation of Cascading Blackouts

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Rare Events in Power Grids

Individual Outages in North America, 1984-1997

Power-tailed Distribution

\[ \Pr\{X > x\} \approx cx^{-a} \]

Adapted from John Doyle. 1999. Complexity and Robustness.
Problems Simulating Rare Events

\[ \gamma \equiv \text{Probability of rare event} \]
\[ \hat{\gamma}_n \equiv \frac{\# \text{replications in which rare event occurs}}{n} \]

Relative Error
\[ RE[\hat{\gamma}_n] \equiv \frac{\sqrt{\text{var}[\hat{\gamma}_n]}}{E[\hat{\gamma}_n]} = \frac{\sqrt{\gamma(1-\gamma)/n}}{\gamma} \approx \frac{1}{\sqrt{n\gamma}} \]

Computer time to simulate rare events can be prohibitive

<table>
<thead>
<tr>
<th>Rare Event Probability $\gamma$</th>
<th>Simulation Runs $n$</th>
<th>Required Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-3}$</td>
<td>$10^7$</td>
<td>16.7 minutes</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>$10^9$</td>
<td>1.2 days</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>$10^{11}$</td>
<td>116 days</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>$10^{13}$</td>
<td>31.7 years</td>
</tr>
</tbody>
</table>

Time Required to Achieve 1% Relative Error
(Assumes 1,000 replications per second)
Splitting

Splitting is a technique to improve the efficiency of rare-event simulation

Optimal Splitting

Problem
Minimize: $\text{var}[\hat{\gamma}]$
such that: $b_1n_1 + \ldots + b_mn_m \leq T$

Solution
Assuming probability of advancing from level $j-1$ to level $j$ does not depend on starting state from level $j-1$, the optimal allocation satisfies:

$n_i b_i \left(1 + \frac{n_ip_i}{1-p_i}\right) = n_2 b_2 \left(1 + \frac{n_2p_2}{1-p_2}\right) = \ldots = n_m b_m \left(1 + \frac{n_mp_m}{1-p_m}\right)$

$n_i = \text{number of runs for stage } i$
$b_i = \text{average computing time for stage-}i \text{ simulation}$
$p_i = \text{probability of advancing from level } i-1 \text{ to level } i$

Splitting: Multiple Designs

Problem
Maximize: \( \min_{n_{ij}} \Pr\{\hat{Y}_1 < \hat{Y}_2, \hat{Y}_1 < \hat{Y}_3, \ldots, \hat{Y}_1 < \hat{Y}_n\} \)

such that: \( \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} n_{ij} = T \)

\( b_{ij} \) = average time to simulate design-\( i \) stage-\( j \)
\( n_{ij} \) = number of runs for design-\( i \) stage-\( j \)
\( p_{ij} \) = prob. of advancing from level \( j-1 \) to \( j \), design \( i \)
\( T \) = computing budget
\( \hat{Y}_i \) = estimator for rare-event probability, design \( i \)

Solution (2 designs): Optimal allocation satisfies:

\[
\frac{n_{ij}^2 b_{ij} p_{ij}}{\gamma_i^2 (1 - p_{ij}) \prod_{r \neq j} \left( 1 + \frac{1 - p_{ir}}{n_{ir} p_{ir}} \right)} = \frac{n_{kl}^2 b_{kl} p_{kl}}{\gamma_k^2 (1 - p_{kl}) \prod_{r \neq l} \left( 1 + \frac{1 - p_{kr}}{n_{kr} p_{kr}} \right)}
\]

Application to Power Grids

Objective: Use splitting to improve simulation efficiency in estimating rare-event probabilities of major outages

System Description:
118 buses
186 branches
91 load sides
54 thermal units

One-line Diagram of IEEE 118-bus Test System
IIT Power Group, 2003

IEEE 118-bus Test System
Blackout Model

Start
Randomly trip one line

Check network connectivity

Match generated power and load

Solve linear power flow equations

Check for line trips

(random step)

End
If no new lines trip

Similar to some models in literature, for example:


Example: Simple Model

- $N$ identical parallel lines connecting two buses
- When a line fails, its load is equally distributed among the remaining lines
- Can be solved analytically as a Markov chain.
- Results obtained for simple network provide insight into application of splitting method for more complex networks

Similar to analytical model in Dobson, Carreras, Newman (2005)
As more lines fail, probability of reaching 100% failure state accelerates.
Splitting: Choice of Levels

- Evenly spaced by distance

- Evenly spaced by probability

  Probability of advancing from one level to the next is approximately the same (cascading effect implies greater spacing at higher levels)

Levels are evenly spaced

Levels are evenly spaced by distance.
Alternate Level Function

- System state: \((m, n)\)
  - \(n\) = \# of presently failed lines
  - \(m\) = \# of failed lines in previous iteration

Levels = states that are (approximately) equally likely to reach rare event
Simulation Efficiency

Objective: Estimate $\gamma = \Pr\{\text{all lines fail}\}$

- Standard simulation inadequate
- Levels evenly spaced by prob. better than evenly spaced
- Modified allocation better than equal-allocation splitting

Prob. of advancing from level $j-1$ to $j$ depends on starting state from level $j-1$ (modified allocation not necessarily optimal)
Example: Mesh Network

Example: 15-bus mesh network
5 generators, 10 loads

Pr\{Blackout \geq n\} vs # failed links n

Blackout accelerates
Decelerates
Simulation Efficiency

Objective: Estimate $\gamma = \Pr\{50 \text{ lines fail}\}$

Prob. of advancing from level $j-1$ to $j$ depends on starting state from level $j-1$ (modified allocation not necessarily optimal)
Example: Grid Network

Each node has load $L$

Blackout accelerates then “dies” out
Objective: Estimate $\gamma = \Pr\{32 \text{ lines fail (out of 40)}\}$

Prob. of advancing from level $j-1$ to $j$ depends on starting state from level $j-1$ (modified allocation not necessarily optimal)
Example: 118-bus System
Line Failures in 20-Line Blackout

Fraction of time line fails in $k$-line blackout

Red = 20-line blackouts
Blue = 4-line blackouts

Alternate level functions: Weight line failures by:
- Power flow through line
- Number lines connected to failed line
- Fraction of time line fails in $k$-line blackout
Summary and Conclusions

• Allocation method for computing budget in rare-event splitting

• Application to model of stochastic cascading line failures
  – Simple analytical network. Choice of levels more important than choice of level function. Modified allocation method provides variance reduction.
  – Alternate models: Cascading nature of blackouts suggests levels with increasing spacing. Modified allocation method generally provides variance reduction