Stable and Efficient Modeling of Anelastic Attenuation in Seismic Wave Propagation

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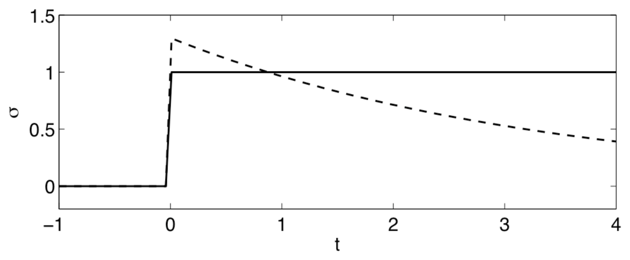
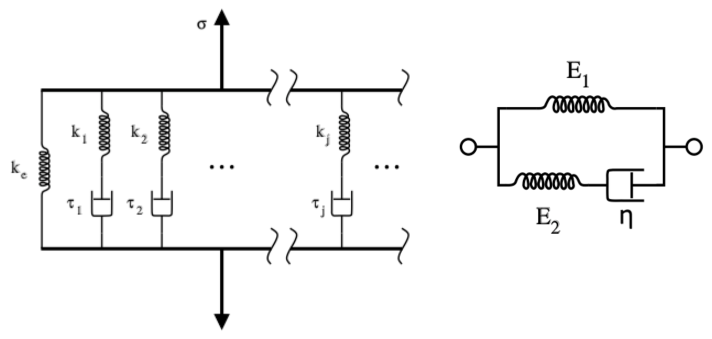
We develop a stable finite difference approximation of the three-dimensional viscoelastic wave equation, based on a generalized Maxwell material model. This visco-elastic material is commonly is used in seismology to approximate a constant-Q absorption band solid. The proposed scheme discretizes the governing equations in second order displacement formulation using 3 memory variables per visco-elastic mechanism, making it significantly more memory efficient than the commonly used first order velocity-stress formulation. The new scheme is a generalization of our energy conserving finite difference scheme for the elastic wave equation in second order formulation. Our main result is a proof that the proposed scheme is energy stable, also for heterogeneous material models. The proof relies on the summation by parts (SBP) property of the discretization. Numerical experiments verify verify the accuracy and stability of the new scheme. Semi-analytical solutions for the LOH.3 layer over half-space problem is used to demonstrate how the number of visco-elastic mechanisms and the grid resolution influence the accuracy. We find that three mechanisms usually are sufficient to make the modeling error smaller than the discretization error.

A generalized Maxwell material is used to approximate a visco-elastic constant-Q absorption band solid in the time-domain

In a visco-elastic material, the stress due to a step function loading relaxes over time



Coupling 'n' standard linear solids in parallel gives a generalized Maxwell material



Corresponding to the two Lamé' parameters in an isotropic elastic material, a generalized Maxwell material is described by two stress relaxation functions

$$\mu(t) = H(t)\left[\mu_0 - \sum_{\nu=1}^{n} \mu_\nu (1 - e^{-\omega_\nu t})\right]$$

$$\lambda(t) = H(t)\left[\lambda_0 - \sum_{\nu=1}^{n} \lambda_\nu (1 - e^{-\omega_\nu t})\right]$$

In frequency space, the visco-elastic shear modulus is defined in terms of the Fourier transform of the stress relaxation function

$$\hat{M}_S(\omega) =: i\omega\hat{\mu}(\omega) = \mu_0 \hat{m}_S(\omega)$$

$$\hat{m}_S(\omega) = 1 - \sum_{\nu=1}^{n} \frac{\beta_\nu(\omega_\nu^2 - i\omega\omega_\nu)}{\omega_\nu^2 + \omega^2} \qquad \beta_\nu = \frac{\mu_\nu}{\mu_0}$$

In seismology, the quality factor 'Q' (and the loss angle δ) is observed to be constant over two decades in frequency.

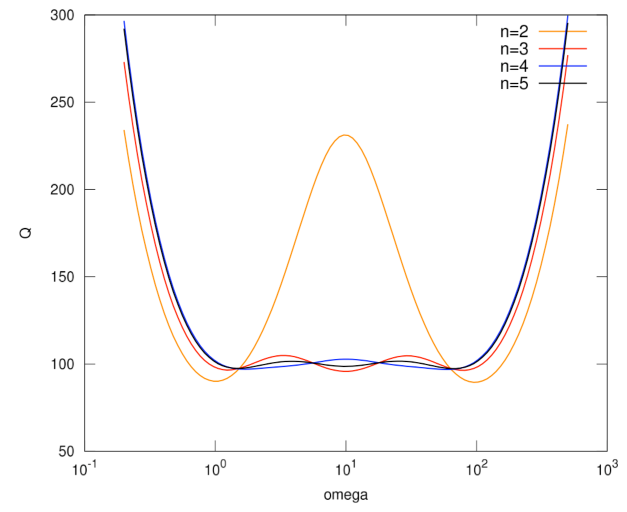
$$Q_S(\omega) =: \frac{\operatorname{Re} \hat{M}_S}{\operatorname{Im} \hat{M}_S} \qquad \tan\delta = 1/Q_S$$

Emmerich and Korn's [1] procedure for determining β_ν:

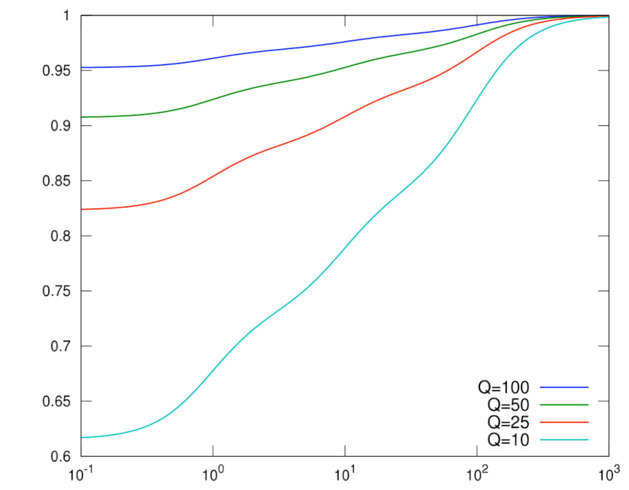
1. **Relaxation frequencies ω_ν, $\nu=1,2,\ldots,n$, logarithmically distributed over $[\omega_{min}, \omega_{max}]$**
2. **Set $Q(\omega)=Q_0$=const. at 2n-1 collocation frequencies, also logarithmically distributed over $[\omega_{min}, \omega_{max}]$**
3. **Solve over-determined linear system for β_ν using least squares**

[1] H. Emmerich and M. Korn. Geophysics, 52(9):1252–1264, 1987.

We plot the actual quality factor as function of scaled frequency, ω/ω_{min} when ω_{max}=100 ω_{min}. Note that n=2 is inadequate, but n=3 gives a much better approximation. Only minor improvements are achieved by increasing 'n' further.



Shear waves and compressional waves attenuate at different rates. A similar procedure is used to determine λ_ν based on the quality factor Q_P for compressional waves.



The visco-elastic material is dispersive, i.e., the phase velocity depends on frequency. The un-relaxed shear modulus μ_0 can be determined after specifying the phase velocity for shear waves, c_s, at a reference frequency ω_r.

$$c_s^2(\omega) = \frac{\mu_0 |m_s(\omega)|}{\rho\cos^2(\delta/2)}$$

Using the second order formulation, we derive sufficient conditions on the material properties through an energy estimate

We use memory variables based on the history of the displacement (instead of the strain) to express the stress tensor as function of the strain tensor.

$$\mathcal{T} = \lambda_0(\nabla\cdot\mathbf{u})\mathcal{I} + 2\mu_0\mathcal{D}(\mathbf{u}) - \sum_{\nu=1}^{n}\left[\lambda_\nu(\nabla\cdot\bar{\mathbf{u}}^{(\nu)}) + 2\mu_\nu\mathcal{D}(\bar{\mathbf{u}}^{(\nu)})\right] \qquad \mathcal{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$$

$$\bar{\mathbf{u}}^{(\nu)}(\mathbf{x},t) = \omega_\nu \int_{-\infty}^{t} \mathbf{u}(\mathbf{x},\tau)e^{-\omega_\nu(t-\tau)}\,d\tau \qquad \frac{1}{\omega_\nu}\frac{\partial\bar{\mathbf{u}}^{(\nu)}}{\partial t} + \bar{\mathbf{u}}^{(\nu)} = \mathbf{u}$$

The visco-elastic wave equation governs the evolution of the displacement.

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = \nabla\cdot\mathcal{T} + \mathbf{F} = \mathbf{L}(\lambda_0,\mu_0)\mathbf{u} - \sum_{\nu=1}^{n}\mathbf{L}(\lambda_\nu,\mu_\nu)\bar{\mathbf{u}}^{(\nu)} + \mathbf{F}$$

$$\mathbf{L}(\lambda,\mu)\mathbf{u} =: \nabla(\lambda(\nabla\cdot\mathbf{u})) + \nabla\cdot(2\mu\mathcal{D}(\mathbf{u}))$$

Integration by parts shows that the spatial operator is self-adjoint wrt the L_2 scalar product.

$$(\mathbf{v}, \mathbf{L}(\lambda_\nu,\mu_\nu)\mathbf{u}) = -S_\nu(\mathbf{v},\mathbf{u}) + B_\nu(\mathbf{v},\mathbf{u})$$

$$S_\nu(\mathbf{v},\mathbf{u}) = (\nabla\cdot\mathbf{v}, \lambda_\nu\nabla\cdot\mathbf{u}) + \int_\Omega 2\mu_\nu\mathcal{D}(\mathbf{v}):\mathcal{D}(\mathbf{u})\,d\Omega$$

The boundary terms cancel for Dirichlet or free-surface b.c.

$$B_\nu(\mathbf{v},\mathbf{u}) = \int_\Gamma \mathbf{v}\cdot[(\lambda_\nu\nabla\cdot\mathbf{u})\mathbf{n} + 2\mu_\nu\mathcal{D}(\mathbf{u})\mathbf{n}]\,d\Gamma$$

We define the visco-elastic energy according to:

$$e(t) = \|\sqrt{\rho}\mathbf{u}_t\|^2 + S_0(\mathbf{u},\mathbf{u}) - \sum_{\nu=1}^{n}S_\nu(\mathbf{u},\mathbf{u}) + \sum_{\nu=1}^{n}S_\nu(\mathbf{u}-\bar{\mathbf{u}}^{(\nu)}, \mathbf{u}-\bar{\mathbf{u}}^{(\nu)})$$

Since S_ν is linear in λ_ν and μ_ν, we have

$$S_0(\mathbf{u},\mathbf{u}) - \sum_{\nu=1}^{n}S_\nu(\mathbf{u},\mathbf{u}) = (\nabla\cdot\mathbf{u}, \tilde{\lambda}\nabla\cdot\mathbf{u}) + \int_\Omega 2\tilde{\mu}\mathcal{D}(\mathbf{u}):\mathcal{D}(\mathbf{u})\,d\Omega$$

<u>Theorem 1:</u>
Assume that the material data satisfies:

$$\tilde{\lambda} =: \lambda_0 - \sum_{\nu=1}^{n}\lambda_\nu \geq \tilde{\lambda}_{min} > 0, \qquad \tilde{\mu} =: \mu_0 - \sum_{\nu=1}^{n}\mu_\nu \geq \tilde{\mu}_{min} > 0$$

$$\rho \geq \rho_{min} > 0, \qquad \lambda_\nu \geq \lambda_{min} > 0, \qquad \mu_\nu \geq \mu_{min} > 0$$

Then, the solution of the visco-elastic wave equation with F=0, subject to Dirichlet or free-surface b.c., has non-increasing energy.

$$e(t) \leq e(0), \qquad \rho_{min}\|\mathbf{u}_t\|^2 + 2\tilde{\mu}_{min}\|\mathcal{D}(\mathbf{u})\|^2 + \tilde{\lambda}_{min}\|\nabla\cdot\mathbf{u}\|^2 \leq e(t)$$

We get an energy stable scheme by using summation by parts operators in space and a hybrid Leap-Frog / Crank-Nicholson scheme for the memory variables

We discretize the spatial operator using a 2nd order accurate finite difference scheme that satisfies a summation by parts identity in a weighted scalar product.

$$\rho\frac{\mathbf{u}^{m+1} - 2\mathbf{u}^m + \mathbf{u}^{m-1}}{\Delta t^2} = \mathbf{L}_h(\lambda_0,\mu_0)\mathbf{u}^m - \sum_{\nu=1}^{n}\mathbf{L}_h(\lambda_\nu,\mu_\nu)\bar{\mathbf{u}}^{(\nu),m} + \mathbf{F}^m$$

$$(\mathbf{v}, \mathbf{L}_h(\lambda_\nu,\mu_\nu)\mathbf{u})_h = -S_\nu^{(h)}(\mathbf{v},\mathbf{u}) + B_\nu^{(h)}(\mathbf{v},\mathbf{u})$$

The boundary terms cancel for Dirichlet or free-surface b.c. An energy estimate can be derived for the semi-discrete approximation, using the same technique as the continuous problem.

$$S_\nu^{(h)}(\mathbf{v},\mathbf{u}) = S_\nu^{(h)}(\mathbf{u},\mathbf{v}), \qquad S_\nu^{(h)}(\mathbf{u},\mathbf{u}) \geq 0$$

We discretize the differential equation for the memory variables using a hybrid scheme.

$$\frac{1}{\omega_\nu}\frac{1}{2\Delta t}\left(\bar{\mathbf{u}}^{(\nu),m+1} - \bar{\mathbf{u}}^{(\nu),m-1}\right) + \frac{1}{2}\left(\bar{\mathbf{u}}^{(\nu),m+1} + \bar{\mathbf{u}}^{(\nu),m-1}\right) = \mathbf{u}^m$$

Let $u^{m+1/2}=(u^{m+1} + u^m)/2$ and $D^t_+u^m=(u^{m+1} - u^m)/\Delta t$. Define the discrete energy by

$$e^{m+1/2} = \left\|\sqrt{\rho}D_+^t\mathbf{u}^m\right\|_h^2 + S_0^h(\mathbf{u}^{m+1/2},\mathbf{u}^{m+1/2}) - \sum_{\nu=1}^{n}S_\nu^{(h)}(\mathbf{u}^{m+1/2},\mathbf{u}^{m+1/2})$$

$$-\frac{\Delta t^2}{4}\sum_{\nu=0}^{n}S_\nu^{(h)}(D_+^t\mathbf{u}^m, D_+^t\mathbf{u}^m) + P^{m+1/2}, \qquad P^{m+1/2} \geq 0$$

<u>Theorem 2:</u>
Assume that the material data satisfy the conditions from Theorem 1, and that the time-step satisfies:

$$\Delta t \leq \Delta t_{max} = \frac{2\sqrt{1-\alpha}}{\sqrt{\zeta_{max}}}, \qquad \zeta_{max} = \max_{\mathbf{v}\neq 0}\frac{\sum_{\nu=0}^{n}S_\nu^{(h)}(\mathbf{v},\mathbf{v})}{(\mathbf{v},\rho\mathbf{v})_h}, \qquad 0 < \alpha \ll 1$$

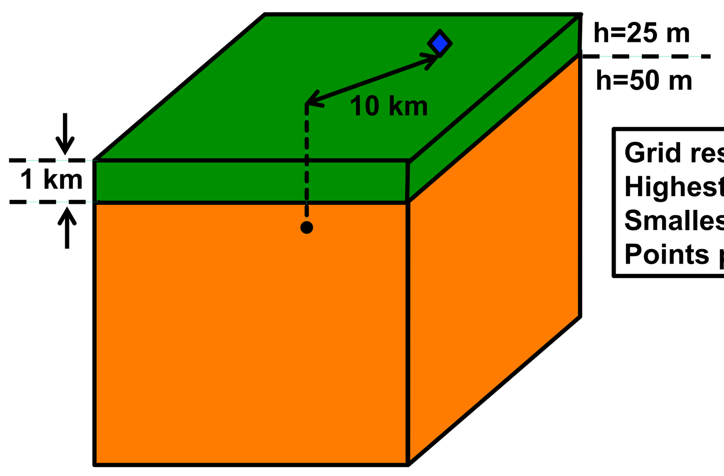
Then, the solution of the discrete visco-elastic wave equation with F=0, subject to Dirichlet or free-surface b.c. has non-increasing discrete energy.

$$e^{m+1/2} \leq e^{m-1/2} \leq \ldots \leq e^{1/2}, \qquad \alpha\|\sqrt{\rho}D_+^t\mathbf{u}^m\|_h^2 \leq e^{m+1/2}$$

<u>Remark:</u> It is also possible to formulate the visco-elastic wave equation in terms of memory variables for the strain tensor. While this formulation is equivalent for the continuous equations, it requires 6 dependent variables per mechanism instead of 3. Furthermore, it is not known if those equations can be discretized such that sufficient conditions for stability can be established.

Numerical experiments show that n=3 mechanisms often make the modeling error smaller than the discretization error

The LOH.3 test problem [2]:
Top layer: C_p=4 km/s, C_s=2 km/s, ρ=2.6 Mg/m^3, Q_P=120, Q_S=40
Half-space: C_p=6 km/s, C_s=3.464 km/s, ρ=2.7 Mg/m^3, Q_P=155.9, Q_S=69.3
Phase velocities at 2.5 Hz, , ω_{min}=0.15 Hz, ω_{max}=15 Hz
Source at 2 km depth, $M_{xy}=10^{18}$ Nm, Gaussian time function, f_0=3.18 Hz

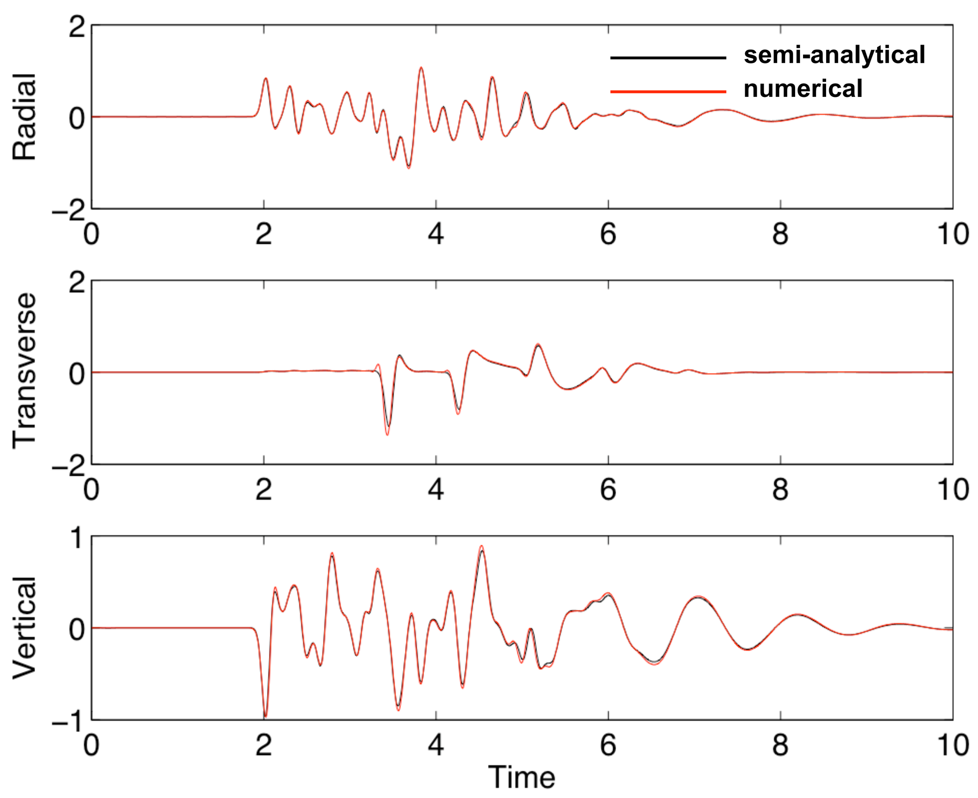


Grid resolution:
Highest frequency, f_{up}= 7.95 Hz
Smallest wave length, λ_{min}=251 m
Points per wave length, P=10.04

3.49×10^8 grid points, 2560 time steps

n	Norm of error @ 10km	CPU-time (512 cores)
2	1.31e-1	25 min, 30 sec.
3	4.84e-2	31 min., 14 sec.
4	5.09e-2	36 min, 7 sec.

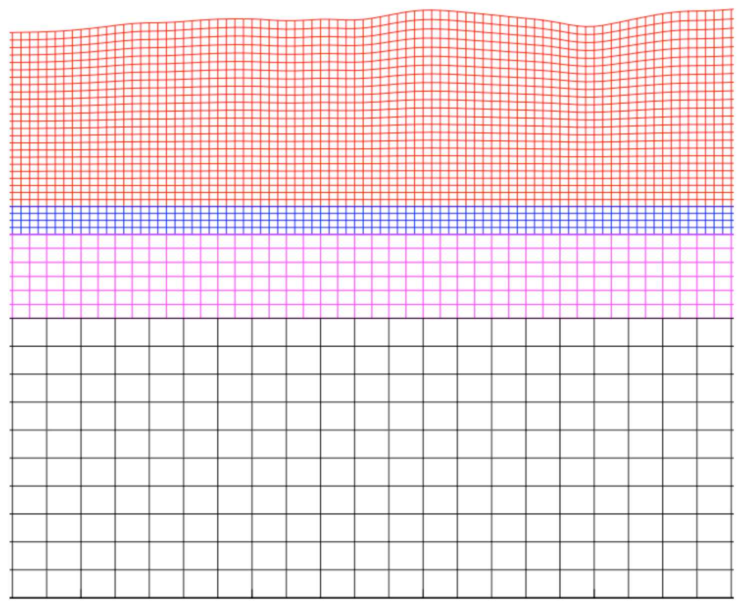
Motion at receiver 10 km from epicenter



[2] S. M. Day, et al. Test of 3D elastodynamic codes: Lifelines program task 1A02. Pacific Earthquake Engineering Center, 2003.

The visco-elastic modeling has been generalized to curvilinear grids and mesh refinement with hanging nodes, and is part of WPP version 2.1

- **Second order formulation:**
  - Less memory than a 1st order velocity/stress formulation
  - No worries about Saint-Venant compatibility conditions
- **Conservative finite difference discretization**
  - Summation by parts (SBP) principle
  - Stable long-time simulations in heterogeneous media with free surfaces
  - Not the standard staggered grid FD
- **Easy grid generation with a composite grid approach**
  - Curvilinear boundary conforming mesh near topography surface
  - Coarser and coarser Cartesian meshes away from surface
  - Energy conserving with hanging nodes
- **Kinematic source model**
  - Moment tensor & point force source terms with many time functions
- **MPI for parallel runs**
  - Tested on up to 32,768 cores
- **Extensively verified**
  - Method of manufactured solutions
  - Lamb's problem
  - Layer over half-space problems
  - Comparisons with other codes
- **Project website and software download**
  - computation.llnl.gov/casc/serpentine



More information in our paper:
[3] N.A. Petersson and B. Sjogreen, Stable and Efficient Modeling of Anelastic Attenuation in Seismic Wave Propagation, *Comm. Comput. Phys. (to appear)*, (2011).

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