A Boundary Perturbation Method for Recovering Interface Shapes in Layered Media

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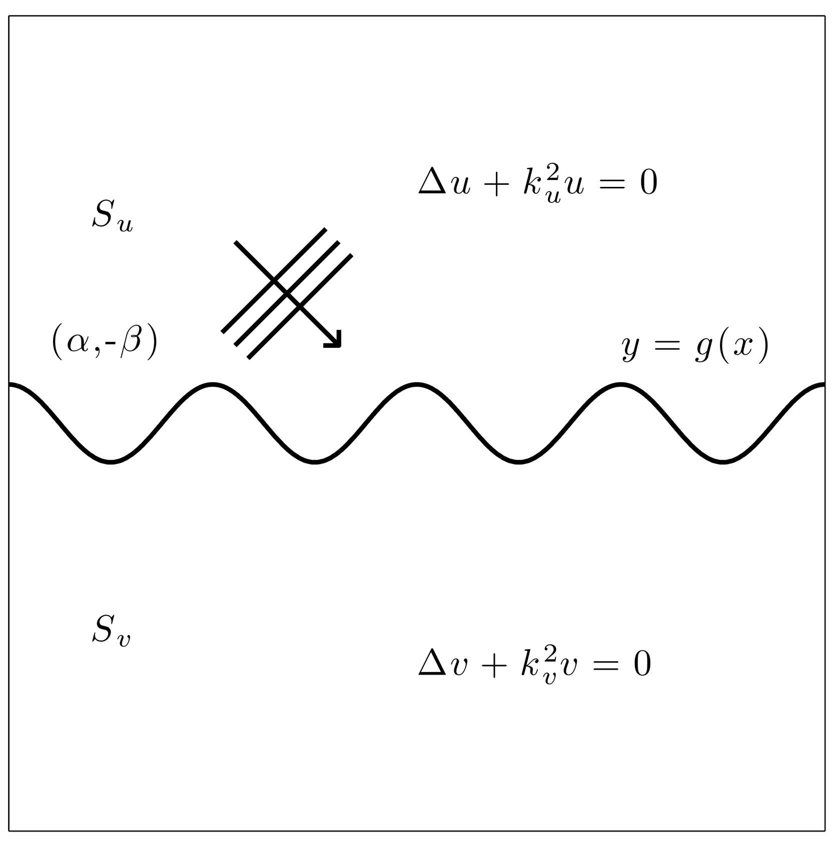
Introduction

- The scattering of linear acoustic radiation by a periodic layered structure is a fundamental model in the geosciences as it closely approximates the propagation of pressure waves in the earth's crust.
- We describe new algorithms for:
  1. The **forward problem** of prescribing incident radiation and, given known structure, determining the scattered field,
  2. The **inverse problem** of approximating the form of the structure given prescribed incident radiation and measured scattered data.
- Each of these algorithms is based upon a novel statement of the problem in terms of boundary integral operators (Dirichlet–Neumann operators), and a Boundary Perturbation algorithm (the Method of Operator Expansions) for their evaluation.

Model



- Consider scattering of an acoustic plane–wave
$$\tilde{u}^i = e^{i\alpha x - i\beta_u y - i\omega t} = u^i e^{-i\omega t}$$
inside a two–layered medium with interface shaped by $y = g(x)$ and velocities $\{c_u, c_v\}$.
- **Forward problem:** Find scattering returns given geometry.
- **Inverse problem:** Recover geometry given far–field data.

- The governing equations for the *scattered* pressure fields $\{u, v\}$ are
$$\begin{aligned}
&\Delta u + k_u^2 u = 0 && y > g(x)\\
&\mathscr{B}_u\{u\} = 0 && y \to \infty\\
&\Delta v + k_v^2 v = 0 && y < g(x)\\
&\mathscr{B}_v\{v\} = 0 && y \to -\infty\\
&u - v = \zeta := -e^{i\alpha x - i\beta_u g} && y = g\\
&\partial_N u - \partial_N v = \psi := (i\beta_u + i\alpha(\partial_x g))e^{i\alpha x - i\beta_u g} && y = g,
\end{aligned}$$
where the wavenumbers are $k_j = \omega / c_j$, the upward–pointing normal is $N := (-\partial_x g, 1)^T$, and the operators $\mathscr{B}_j$ enforce the *Outgoing Wave Conditions.*
- **Rayleigh's Solutions** tell us much of what we need to know to solve this problem:
$$u(x,y) = \sum_{p=-\infty}^{\infty} a_p e^{i\alpha_p x + i\beta_{u,p} y}, \quad v(x,y) = \sum_{p=-\infty}^{\infty} b_p e^{i\alpha_p x - i\beta_{v,p} y},$$
for $y > |g|_\infty$ and $y < -|g|_\infty$, respectively. In these equations
$$\alpha_p := \alpha + (2\pi/d)p, \quad \beta_{j,p} := \begin{cases} \sqrt{k_j^2 - \alpha_p^2} & \alpha_p^2 < k_j^2 \\ i\sqrt{\alpha_p^2 - k_j^2} & \alpha_p^2 > k_j^2 \end{cases}.$$
Notice that these both solve the Helmholtz equation, and are "outgoing" (i.e., bounded at infinity).

Surface Formulation

- Given Dirichlet and *exterior* Neumann traces of u and v
$$U(x) := u(x, g(x)), \quad V(x) := v(x, g(x)),$$
$$U'(x) := -(\partial_N u)(x, g(x)), \quad V'(x) := (\partial_N v)(x, g(x)),$$
integral formulas will give us u and v everywhere.
- Furthermore, if we define **Dirichlet–Neumann Operators (DNOs)**
$$G(g)[U(x)] := U'(x), \quad H(g)[V(x)] := V'(x),$$
then it suffices to find the Dirichlet traces U and V.
- It is not difficult to see that the full system of governing equations simplifies to
$$U - V = \zeta, \quad -G[U] - H[V] = \psi.$$
- Solving for V in terms of U we find the *single* equation:
$$(G+H)[U] = -\psi + H[\zeta]. \tag{1}$$
- With a view to the inverse problem, we note that U is an inconvenient unknown: It is defined *on* the interface. It is equivalent to solve for the "far–field data"
$$\tilde{u}(x) := u(x, a), \quad a > |g|_\infty.$$

- To use (1) with $\tilde{u}$ we introduce the "Backward Propagator Operator" L:
$$L(g)[\tilde{u}(x)] := U(x),$$
which is, evidently, **VERY POORLY CONDITIONED!**
- Our final equation becomes
$$0 = Q(g)[\tilde{u}] := (G+H)[L[\tilde{u}]] + \psi - H[\zeta]. \tag{2}$$

The Forward Problem

- First, we introduce a *perturbative* approach to solving (2). It can be shown that, if $g(x) = \varepsilon f(x)$, f is *sufficiently smooth* (e.g., C^2), and ε is *sufficiently small* then
$$\{\zeta, \psi, G, H, L\}(\varepsilon) = \sum_{n=0}^{\infty} \{\zeta_n, \psi_n, G_n, H_n, L_n\}\varepsilon^n$$
i.e., are all *analytic* in the boundary perturbation.
- It can be shown *a posteriori* that
$$\tilde{u}(x;\varepsilon) = \sum_{n=0}^{\infty} \tilde{u}_n(x)\varepsilon^n$$
so that a natural method to approximate $\tilde{u}$ is by its truncated Taylor series with terms
$$\tilde{u}_n = -Q_0^{-1}\left[\sum_{m=0}^{n-1} Q_{n-m}[\tilde{u}_m]\right],$$
where, e.g.,
$$Q_0 = (G_0 + H_0)L_0, \quad Q_0^{-1} = L_0^{-1}(G_0 + H_0)^{-1}.$$
- To give a flavor for the algorithm we note that
$$G_0[\xi] = G_0\left[\sum_{p=-\infty}^{\infty} \hat{\xi}_p e^{i\alpha_p x}\right] = \sum_{p=-\infty}^{\infty} (-i\beta_{u,p})\hat{\xi}_p e^{i\alpha_p x} =: -(i\beta_{u,D})\xi,$$
$H_0 = -i\beta_{v,D}$, and $L_0 = \exp(-i\beta_{u,D} a)$. We note that L_0^{-1} is **exponentially smoothing** while L_0 is **ill–conditioned**.

The Inverse Problem

- For the inverse problem we take a different point of view towards (2): We specify $\tilde{u}$ and seek g.
- Once again, we use the *analytic* nature of the surface data $\{\zeta, \psi\}$ and operators $\{G, H, L\}$ to express
$$Q(g)[\tilde{u}] = Q(\varepsilon f)[\tilde{u}] = \sum_{n=0}^{\infty} \varepsilon^n Q_n(f)[\tilde{u}].$$
- **Linear Model:** Using this expansion we see that (2) becomes
$$Q_0[\tilde{u}] + Q_1(g)[\tilde{u}] = \mathscr{O}(g^2). \tag{3}$$
Truncating this at order two, we derive a Linear Model:
$$\tilde{g}^0 = -\{Q_1(\cdot)[\tilde{u}]\}^{-1} Q_0[\tilde{u}].$$
- **Nonlinear Model:** Alternatively, we can cast (2) as
$$Q_0[\tilde{u}] + Q_1(g)[\tilde{u}] + \sum_{n=2}^{N} Q_n(g)[\tilde{u}] = \mathscr{O}(g^{N+1}).$$
Truncating at order $N+1$ we derive a Nonlinear Model:
$$\tilde{g}^{k+1} = -\{Q_1(\cdot)[\tilde{u}]\}^{-1}\left(Q_0[\tilde{u}] + \sum_{n=2}^{N} Q_n(\tilde{g}^k)[\tilde{u}]\right), \tag{4}$$
where we begin with $\tilde{g}^0$ from (3), and proceed until $\left\|\tilde{g}^{k+1} - \tilde{g}^k\right\| < \tau$.

Results

Consider the analytic profile $y = \varepsilon e^{\cos(2x)}$, and the Linear (3) and Nonlinear (4) models for reconstruction. With physical parameters
$$\alpha = 0, \quad \beta_u = 1.1, \quad \beta_v = 5.5, \quad d = 2\pi, \quad a = 1,$$
and numerical parameters $N_x = 32$, $N = 4$, $\tau = 10^{-8}$ we achieve the following results.

ε	Abs. L^∞ Error	Rel. L^∞ Error
0.001	3.40341×10^{-6}	0.00125205
0.002	1.35404×10^{-5}	0.00249062
0.003	3.02975×10^{-5}	0.00371528
0.004	5.35726×10^{-5}	0.00492706
0.005	8.32629×10^{-5}	0.00612614
0.006	0.00011928	0.00731342
0.007	0.000161528	0.008489
0.008	0.000209926	0.00965343
0.009	0.000264389	0.010807
0.01	0.000324838	0.0119501

ε	Num. Iter.	Abs. L^∞ Error	Rel. L^∞ Error
0.001	4	1.21923×10^{-9}	4.48531×10^{-7}
0.002	5	1.05361×10^{-9}	1.938×10^{-7}
0.003	6	1.50681×10^{-9}	1.84775×10^{-7}
0.004	7	3.99985×10^{-9}	3.67865×10^{-7}
0.005	8	7.41919×10^{-9}	5.45873×10^{-7}
0.006	9	2.03556×10^{-8}	1.24807×10^{-6}
0.007	10	4.26912×10^{-8}	2.2436×10^{-6}
0.008	11	8.29894×10^{-8}	3.81626×10^{-6}
0.009	12	1.50113×10^{-7}	6.13594×10^{-6}
0.01	13	2.56547×10^{-7}	9.43782×10^{-6}