M-Adaptation

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I. Mimetic Finite Difference Method

THE mimetic finite difference (MFD) method mimics essential properties of differential equations such as conservation laws, symmetries, maximum principles, and fundamental identities and theorems of a vector and tensor calculus on polygonal and generalized polyhedral meshes.

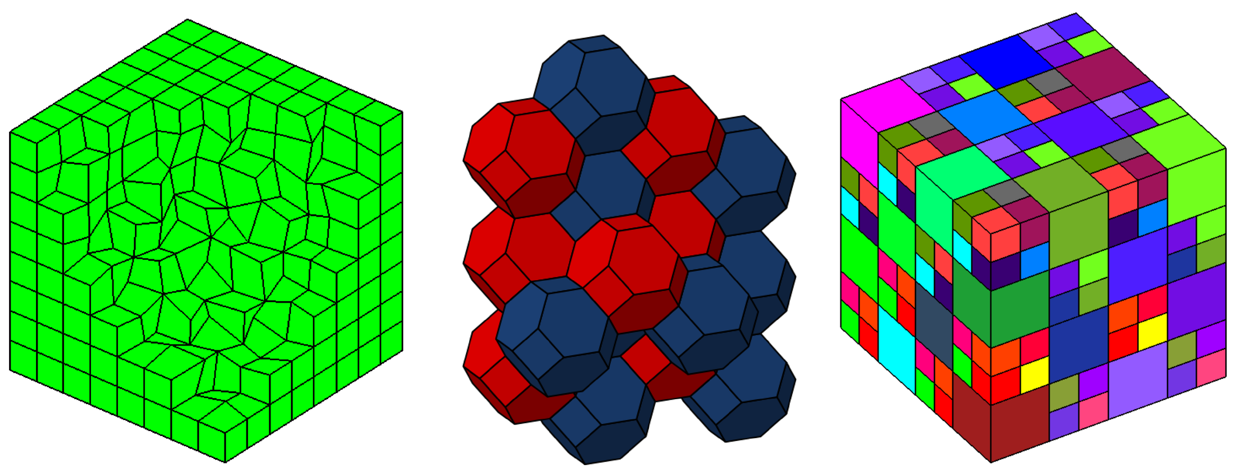


Figure 1: *Admissible meshes: a distorted logically cubic mesh with non-planar faces, a Voronoi-type polyhedral mesh, a locally refined (AMR) mesh with degenerate cells.*

THE MFD method works with discrete fields defined at various mesh objects (nodes ($\mathcal{N}$), edges ($\mathcal{E}$), faces ($\mathcal{F}$) or cells ($\mathcal{C}$)) and the first principles to build the *primary* operators

$$\mathcal{GRAD}^h : \mathcal{N} \to \mathcal{E}, \quad \mathcal{CURL}^h : \mathcal{E} \to \mathcal{F}, \quad \mathcal{DIV}^h : \mathcal{F} \to \mathcal{C}.$$

The dual operators,

$$\widetilde{\mathcal{DIV}}^h : \mathcal{E} \to \mathcal{N}, \quad \widetilde{\mathcal{CURL}}^h : \mathcal{F} \to \mathcal{E}, \quad \widetilde{\mathcal{GRAD}}^h : \mathcal{C} \to \mathcal{F},$$

are derived from discrete integration by parts formulas, e.g.

$$[\mathbf{u}_h, \widetilde{\mathcal{GRAD}}^h(p_h)]_{\mathcal{F}} = -[\mathcal{DIV}^h(\mathbf{u}^h), p_h]_{\mathcal{C}}$$

where

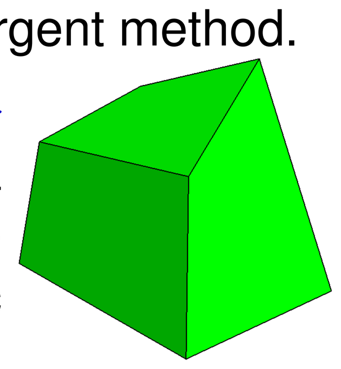
$$[\mathbf{u}_h, \mathbf{v}_h]_{\mathcal{F}} = \mathbf{U}^T \mathbb{M}_{\mathcal{F}} \mathbf{V}, \qquad [p_h, q_h]_{\mathcal{C}} = \mathbf{P}^T \mathbb{M}_{\mathcal{C}} \mathbf{Q}$$

are L^2-type inner products on a space $\mathcal{F}$ of face-based fields and a space $\mathcal{C}$ of cell-based fields, respectively. By construction, the dual and primary operators satisfy the discrete identities, e.g. $\widetilde{\mathcal{CURL}}^h \widetilde{\mathcal{GRAD}}^h = 0$, $\mathcal{CURL}^h \mathcal{GRAD}^h = 0$.

L^2-type inner products

ONLY properly defined symmetric and positive definite matrices $\mathbb{M}_{\mathcal{F}}$ and $\mathbb{M}_{\mathcal{C}}$ result in a convergent method.

The MFD method assembles matrix $\mathbb{M}_{\mathcal{F}}$ from elemental matrices $\mathbb{M}_{\mathcal{F},E}$, similar to a finite element method. However, construction of the elemental matrices is algebraic [2], e.g.



$$\mathbb{M}_{\mathcal{F},E} = \mathbb{M}_{\mathcal{F},E}^{consistency} + \mathbb{M}_{\mathcal{F},E}^{stability}.$$

- The matrix $\mathbb{M}_{\mathcal{F},E}^{consistency}$ guarantees consistency of the discretization and comes from a local patch test.
- The matrix $\mathbb{M}_{\mathcal{F},E}^{stability}$ is non-unique and ensures stability of the MFD method. For a cubic mesh and the lowest order discretization, each 6×6 matrix $\mathbb{M}_{\mathcal{F},E}^{stability}$ is defined by 6 parameters, and parameters may vary from cell to cell.

The matrix $\mathbb{M}_{\mathcal{F},E}$ is the first-order quadrature rule for the L^2-type inner product:

$$\mathbf{U}_E^T \mathbb{M}_{\mathcal{F},E} \mathbf{V}_E \approx \int_E \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}x.$$

H^1-type inner products

IN acoustic problems, an elemental matrix $\mathbb{A}_{\mathcal{N},E}$ must represent accurately an H^1-type inner product:

$$\mathbf{P}_E^T \mathbb{A}_{\mathcal{N},E} \mathbf{Q}_E \approx \int_E \nabla \mathbf{p} \cdot \nabla \mathbf{q} \, \mathrm{d}x.$$

The mimetic method builds this matrix algebraically using again a consistency argument. The final decomposition has the typical mimetic form [1]:

$$\mathbb{A}_{\mathcal{N},E} = \mathbb{A}_{\mathcal{N},E}^{consistency} + \mathbb{A}_{\mathcal{N},E}^{stability}.$$

- In the case of simplicial meshes, the MFD method reduces to the finite element method.
- The algebraic construction is a powerful tool for building high-order C^1-continuous discretizations for thin plates problems.
- The matrix $\mathbb{A}_{\mathcal{N},E}^{stability}$ is non-unique and is defined by a few parameters (10 for a cube). Number of parameters grows quadratically with the number of vertices in cell E.

II. M-Adaptation

OBJECTIVE of the m-adaptation is to solve the multi-parameter optimization problem, i.e. to find the best member of the MFD family of discretization methods for a given problem.

Step 1. Formulate a physical or mathematical optimization criterion.

Step 2. Reduce the global optimization problem to a set of local once.

Step 3. Solve the local problems.

Acoustic wave propagation

$$p_{tt} = c^2 \Delta p.$$

The physical optimization criterion is minimization of numerical anisotropy, i.e. reduction of variation of wave speed between different directions.

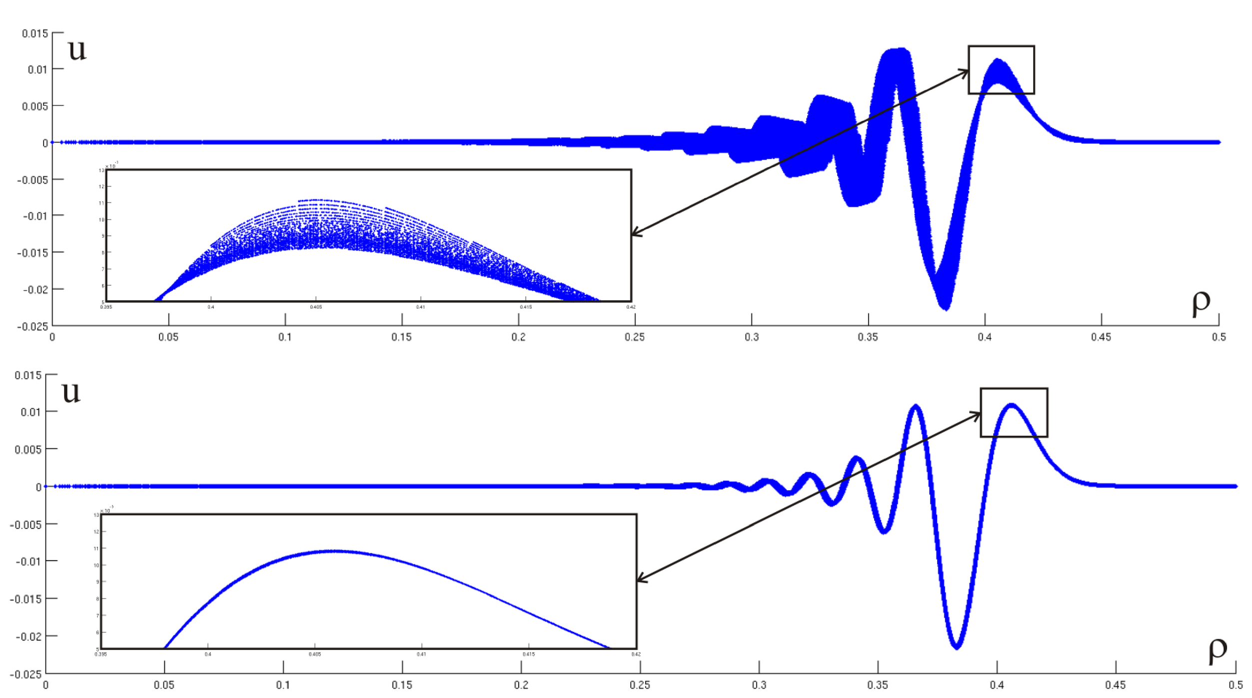


Figure 2: *Amplitude of a spherical wave at nodes of a cubic mesh as a function of radius: FE (top) and optimized MFD (bottom) snapshots after the wave has traveled 15 mean wavelengths; $\lambda_{mean}/h = 10$.*

In each cubic cell, we use the same values for 6 available parameters; thus, reducing the global optimization problem to a single local one.

Steady-state Darcy flow in saturated medium

$$\mathbf{u} = -\mathbb{K}\nabla \mathbf{p}, \quad \mathrm{div}\,(\mathbf{u}) = Q.$$

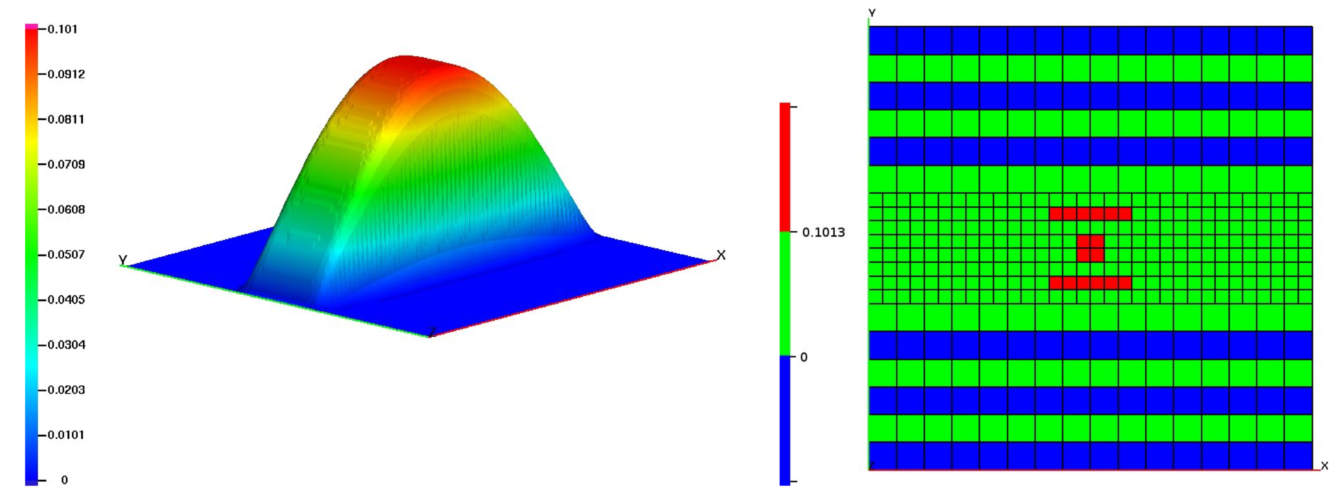
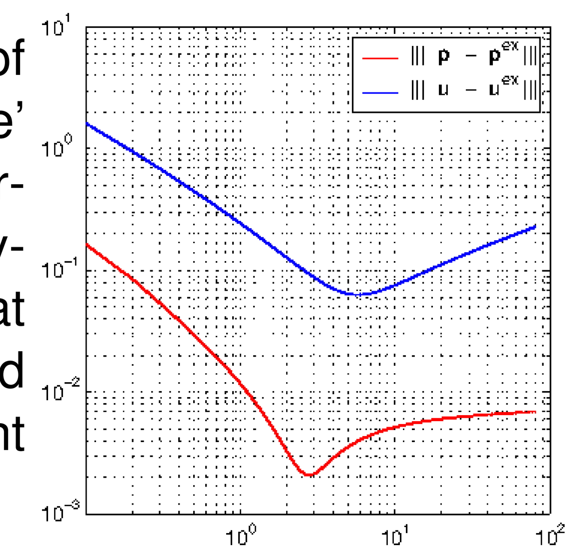


Figure 3: *Left: analytical solution with a discontinuous tensor $\mathbb{K}$. Right: undershoots (blue cells) and overshoots (red cells) in the discrete pressure field p_h calculated with the original MFD method.*

The mathematical optimization criterion is the M-matrix property for matrix $\mathbb{M}_{\mathcal{F},E}^{-1}$ which guarantees the discrete maximum principle.

	Original MFD method			Monotone MFD method		
h	$\min_{\mathbb{P}} p_h$	$\max_{\mathbb{P}} p_h$	ε_2^p	$\min_{\mathbb{P}} p_h$	$\max_{\mathbb{P}} p_h$	ε_2^p
$1/16$	-6.203e-03	1.055e-01	3.62e-03	2.879e-12	1.012e-01	8.96e-04
$1/32$	-4.219e-03	1.013e-01	1.95e-03	8.927e-16	1.013e-01	9.57e-04
$1/64$	-1.418e-13	1.013e-01	7.03e-04	2.172e-18	1.013e-01	5.26e-04
$1/128$	-7.573e-14	1.013e-01	1.99e-04	7.343e-20	1.013e-01	1.81e-04
$1/256$	9.855e-21	1.013e-01	6.66e-05	9.928e-21	1.013e-01	4.98e-05

Study of a 1-parameter family of MFD methods shows that the 'free' parameter can vary over two orders in magnitude without destroying quality of the solution. Note that error minima for the velocity $\mathbf{u}$ and pressure p correspond to different values of the parameter.



III. Future directions

FUTURE development of MFD methods is aligned with demands of various LANL-wide and DOE-wide projects:

- Linear and non-linear monotone MFD methods: ASCEM project of the Office of Environmental Management.
- General MFD methods: Lagrangian hydrocode of the ASC Program, LANL.
- Arbitrary-order MFD methods: modeling of elastic wave propagation in Earth.

References

[1] V.Gyrya, K.Lipnikov. M-adaptation method for solving acoustic wave equation on rectangular meshes. *Int. J. Numer. Meth. Eng.*, under review, 2011.

[2] K.Lipnikov, G.Manzini, D.Svyatskiy. Analysis of the monotonicity conditions in the mimetic finite difference method for elliptic problems. *J. Comp. Phys.*, V.230, pp.2620–2642, 2011.