Infeasible Constraint-Reduced Methods for Quadratic and Semidefinite Optimization

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Introduction

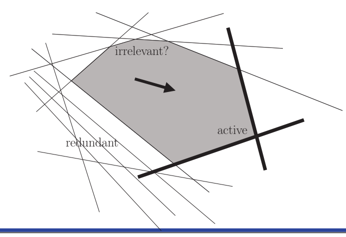
We consider the (dual) problem

$$\max f(y) \quad s.t. \quad A^* y \preceq c, \qquad (D)$$

where $A \in \mathbb{R}^{m \times n}$ and we assume $m \ll n$.

Constraint Reduction

We expect many of the constraints in unbalanced ($m \ll n$) problems are redundant or at least not strictly necessary.



Quadratic optimization: allowing infeasible initial points (M. He)

Problem statement and main idea

- We consider quadratic problems of (D).
  cost function: $f(y) = b^T y - 0.5 y^T H y$, constraints: $A^T y \leq c$.
- It may be hard to find a strictly dual feasible point.
- A remedy: introduce an ℓ_1 penalty function.

$$\max_{y,s} \quad f(y) + \rho \sum_i s_i \qquad \text{s.t.} \quad A^T y - s \leq c, \; s \geq 0 \qquad (D_\rho)$$

where $\rho > 0$ is the penalty parameter.

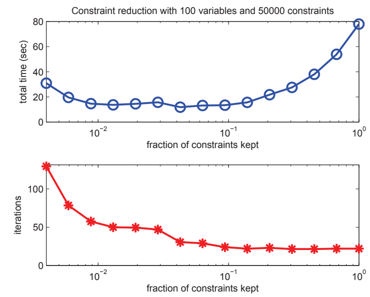
- Penalty function is *exact*: for ρ large enough, solutions to (D_ρ) are solutions to (D).
- Proposed parameter adjustment scheme: for each iteration, increase ρ when (D_ρ) is deemed to be unbounded or when the solution of (D_ρ) is deemed to be infeasible to (D).

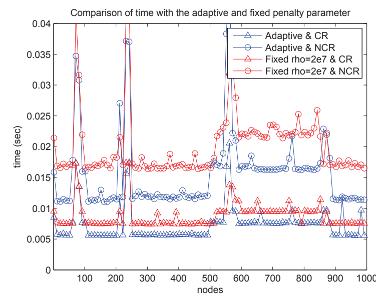
Theoretical Results

- We develop a constraint-reduced primal-dual affine-scaling interior point algorithm, following that of Jung, O'Leary and Tits (2010), but with adaptive adjustment of the penalty parameter.
- We prove that the penalty parameter is increased finitely many times.
- We prove that the algorithm is globally convergent to a point that satisfies the optimality conditions for the QP.
- Generalizes the results obtained for the LP case in He and Tits (2011).

Numerical Results

- First figure: Randomly generated problems
  Second figure: Application in Model Predictive Control





Polynomial algorithm for semi-definite optimization (S. Park)

Problem Statement

- Dual SDP:

$$\max_{\boldsymbol{y}} \boldsymbol{b}^T \boldsymbol{y} \quad \text{s.t.} \quad \sum_{i=1}^m y_i \boldsymbol{A}_i + \boldsymbol{Z} = \boldsymbol{C}, \quad \boldsymbol{Z} \succeq \boldsymbol{0},$$

where $\boldsymbol{C}, \boldsymbol{A}_i, \boldsymbol{X}, \boldsymbol{Z} \in \mathcal{S}^n$ and $m \ll p$.
We focus on problems in which the matrices $\boldsymbol{A}_i$ and $\boldsymbol{C}$ are block diagonal:

$$\boldsymbol{A}_i = \begin{bmatrix} \boldsymbol{A}_{i1} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{A}_{ip} \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{C}_p \end{bmatrix}.$$

Throughout our work we assume the Slater condition, so the primal and dual SDP problems have optimal solutions with equal optimal values.

Results

- We develop (Park 2011) a primal-dual predictor-corrector interior point algorithm, following that of Potra and Sheng (2006), except allowing adaptive constraint reduction, which, in this case, means the omission of certain blocks $\boldsymbol{A}_i$ in the formation of the Schur complement matrix.
- We prove that the algorithm is globally convergent to a point that satisfies the optimality conditions for the SDP.
- We prove that the algorithm converges in $O(n \ln(\epsilon_0/\epsilon))$ iterations, the same as the (unreduced) algorithm of Potra and Sheng, where

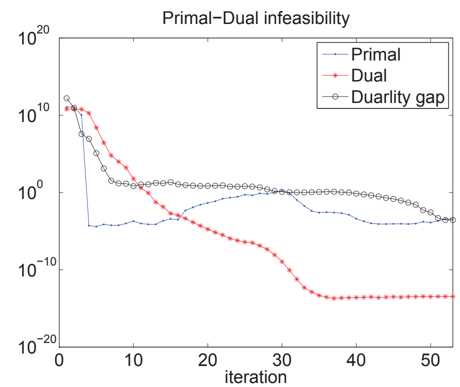
$$\epsilon_0 = \max(\boldsymbol{X}_0 \bullet \boldsymbol{Z}_0, \|\boldsymbol{r}_p^0\|, \|\boldsymbol{r}_d^0\|),$$

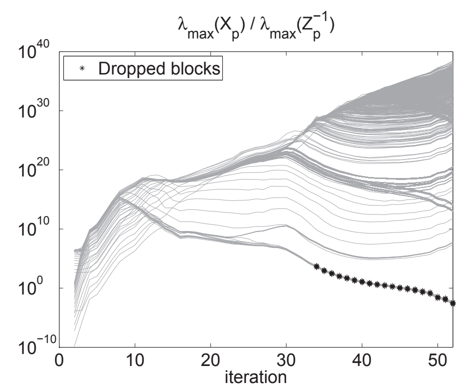
where $\boldsymbol{r}_p^0$ and $\boldsymbol{r}_d^0$ are initial primal and dual residuals, and ϵ is the required tolerance on the optimality conditions. This implies polynomial complexity.

Significance of the Results

- This is the first such complexity result for primal-dual constraint reduction algorithms for any class of problems.
- The algorithm solves, as special cases, any optimization problem that is linear, convex quadratic, convex quadratically constrained, or a second-order cone problem.
- Block diagonal SDPs also arise from relaxations of many important problems with integer variables to problems involving continuous variables. These problems include the maximum binary code problem, the traveling salesperson problem, the kissing number problem, and the quadratic assignment problem.

Application: SDPT3-ConstraintReduced on Schrijver_A(40,15)





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