Multiscale Geometric Dictionaries for Point-cloud Data

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ABSTRACT

We develop a novel geometric multiscale analysis for analyzing intrinsically low dimensional point clouds in high-dimensional space, modeled as samples from a d-dimensional manifold M (in particular a manifold embedded in Rn, in the regime d ≪ n). This type of situation has been recognized as important in various applications, such as the analysis of sounds, images, and gene strings. In this paper we construct data-dependent multiscale dictionaries that aim at efficient encoding and manipulating of the data. Unlike existing constructions, our construction is fast, and are the algorithms that map data points to dictionaries efficiently and correctly. In addition, data points have a guaranteed sparsity in terms of the dictionary elements.

GEOMETRIC WAVELETS

Let \( (M, g) \) be a d-dimensional compact Riemannian manifold isometrically embedded in \( \mathbb{R}^n \), with \( d < n \). Assume we have \( n \) samples drawn i.i.d. from \( M \), according to the natural volume measure \( d\mathcal{V} \) on \( M \). We use such training data to present how to construct geometric wavelets, though our construction easily extends to any point-cloud data, by using locally adaptive dimensions \( d_{ij,k} \) (rather than a fixed d).

Multiscale decomposition.

We start by constructing a multiscale nested partition of \( M \) into dyadic cells \( (C_{j,1}, \ldots, C_{j,2^{j-1}}) \) that satisfy the usual properties of dyadic cubes in \( \mathbb{R}^n \). There is a natural tree \( \mathcal{T} \) associated to the family: for any \( j \in \mathbb{Z} \) and \( k \in J \), let \( \{k' \in J_{j+1} : C_{j,k} \subseteq C_{j,k'} \} \). Also, for \( x \in M \), we denote by \( C_{j,k}(x) \), the unique cell at scale \( j \) that contains \( x \) (similar notation \( P_{j,k} \), \( \Phi_{j,k} \), \( \Psi_{j,k} \) associated to \( C_{j,k} \) are used later).

Multiscale Wavelet.

For every \( j \in \mathbb{Z} \) we define the mean (in \( \mathbb{R}^n \)) as \( \tau_j \mu(x) = E[\mu|C_{j,k}] = \frac{1}{|C_{j,k}|} \int_{C_{j,k}} \mu(x) \, d\mu(x) \) and the covariance by \( \tau_j \Sigma(x) = E[(\mu(x) - \tau_j \mu)(\mu(x) - \tau_j \mu)^\ast] \, d\mu(x) \in C_{j, 2^j} \). Let the rank-\( d \) Singular Value Decomposition (SVD) of \( \tau_j \Sigma(x) \) be \( (\varphi_l, \lambda_l) \), where \( \varphi_l \) is orthonormal and \( \Sigma_l \) is diagonal. The subspace spanned by the columns of \( \varphi_l \) and then translated to pass through \( \tau_j(x) \) is an approximate tangent space to \( M \) at location \( \tau_j(x) \) and scale \( 2^j \). We think of \( \{\varphi_l \}_{l=1}^d \) as a family of geometric scaling functions at scale \( j \). Let \( P_{j, l} \) be the associated affine projection \( P_{j, l}(x) = \varphi_l(x - \tau_j(x)) + \tau_j(x) \) \( \forall x \in C_{j, 1} \).

We define the coarse approximations, at scale \( j \), to the manifold \( M \) and to any point \( y \in M \) as follows:

\[ M_0 = \cup_{k \leq 2^{-j}} P_{j, k}(1_{C_{j, k}}), \quad x_j = P_{j, 0}(x) = P_{j, 1}(x). \]

We demonstrate that the above construction of the multiscale dictionaries is natural in terms of the dictionary elements and coefficients.

In this paper we construct data-dependent dictionaries based on a geometric multiscale analysis (GMDA) of the data, inspired by multiscale techniques in geometric measure theory. These dictionaries are structured in a multiscale fashion; the expansion of a data point on the dictionary elements is guaranteed to have a certain degree of sparsity in both the dictionary elements and the coefficients may be computed by a fast algorithm; the growth of the number of dictionary elements \( I \) as a function of \( \epsilon \) is controlled theoretically and easily to estimate in practice. We call the elements of these dictionaries geometric wavelets, since in some aspects they generalize wavelets from vectors that analyze functions to affine vectors that analyze point clouds.

EXPERIMENTS

![Figure 1](image1.png)

Figure 1: We sample 10,000 points from a 2-D OctahedronWave embedded in \( \mathbb{R}^3 \) and compute the GWT of the data. Bottom left figure shows the wavelet coefficients arranged into the natural tree. The x-axis indicates the points, and the y-axis indicates the scales from coarse (1) to finest (11).

![Figure 2](image2.png)

Figure 2: We apply the GMDA to 224 (cropped) face images from 38 human subjects in fixed frontal pose under varying illumination angles. Bottom row shows approximations of a fixed point (image) and corresponding dictionary elements used.

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References: