Effective Disjunctive Cuts for Convex Mixed Integer Nonlinear Programs

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Abstract

We give a computationally effective procedure for generating disjunctive cutting planes for convex mixed-integer nonlinear programs (MINLPs):

$$\min_{x \in X} \left\{ c^\top x \mid g(x) \leq 0, x_I \in \mathbb{Z}^{|I|} \right\},$$

where $I \subseteq N := \{1, \ldots, n\}$ is the index set of discrete variables, $X = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ is a polyhedral subset of $\mathbb{R}^n$, $g : \mathbb{R}^k \to \mathbb{R}^n$ is a differentiable, convex function of x. Disjunctive cuts are inequalities that are valid for the convex hull of the union of convex sets. Specifically, if

$$R = \{x \in X \mid g(x) \leq 0\}$$

is the feasible region of the continuous relaxation of (MINLP) we seek valid inequalities for the set

$$\mathcal{R}_i^k = \operatorname{conv}\left(\{x \in R \mid x_i \leq k\} \cup \{x \in R \mid x_i \geq k+1\}\right).$$

Inequalities valid for $\mathcal{R}_i^k$ are also valid for (MINLP).

There is an extended formulation of $\mathcal{R}_i^k$ in terms of convex, nonlinear, inequalities using a variable transformation and the perspective function. In 1999, Stubbs and Mehrotra demonstrated the existence of a convex set $\tilde{\mathcal{M}}_i^k \subset \mathbb{R}^{3n+2}$ with the property

$$\operatorname{proj}_x(\tilde{\mathcal{M}}_i^k) = \{x \mid (x, \tilde{y}, \tilde{z}, \lambda, \mu) \in \tilde{\mathcal{M}}_i^k\} = \mathcal{R}_i^k.$$

Furthermore, given a point $\bar{x} \notin \mathcal{R}_i^k$ a disjunctive inequality can be found by solving the projection problem

$$\min_{(x, \tilde{y}, \tilde{z}, \lambda, \mu) \in \tilde{\mathcal{M}}_i^k} \|x - \bar{x}\|. \tag{1}$$

The downside of using (1) to generate inequalities is twofold. First, one must solve a nonlinear program that is twice the size of the original problem in order to generate a valid inequality. Second, the description of the set $\tilde{\mathcal{M}}_i^k$ contains nondifferentiable functions, so numerical difficulties in nonlinear programming software designed for differentiable functions may lead to the generation of invalid inequalities. Stubbs and Mehrotra give computational experience on instances with only up to $n = 30$ variables.

In this work, we describe a computationally effective method for generating disjunctive inequalities for convex MINLPs that relies on solving a sequence of cut-generating linear programs. In the limit, out procedure generates an inequality as strong as the disjunctive inequality of Stubbs and Mehrotra. Using this procedure, we are able to approximately optimize over the rank one lift-and-project closure for a wide range of convex MINLP instances. The results indicate that disjunctive inequalities have the potential to close a significant portion of the integrality gap for convex MINLPs. In addition, we find that using this procedure within a branch-and-cut solver for convex MINLPs yields significant savings in total solution time for many practically-sized instances. Overall, these results suggest that with an effective separation routine, like the one we propose, disjunctive inequalities may be as effective for solving convex MINLPs as they have been for solving mixed-integer linear programs.