A Scalable Numerical Approach to the Solution of MHD Systems

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Abstract

Consider the following model MHD systems:

$$\begin{aligned} u_t + (u \cdot \nabla)u - \mu\Delta u + \nabla p - (\nabla \times B) \times B &= 0, \quad (1) \\ \nabla \cdot u &= 0, \quad (2) \\ B_t - \nabla \times (u \times B) + \eta \nabla \times (\nabla \times B) &= 0, \quad (3) \end{aligned}$$

There are three nonlinear terms in this set of equations, which are the main source of difficulties in numerical simulations. We write:

$$\frac{Du}{Dt} = u_t + (u \cdot \nabla)u \text{ and } \frac{\delta B}{\delta t} = \frac{\partial B}{\partial t} + (u \cdot \nabla)B - (B \cdot \nabla)u = B_t - \nabla \times (u \times B)$$

where $\frac{D}{Dt}$ is the well-known material derivative for the velocity u and $\frac{\delta}{\delta t}$ can be viewed as covariant derivatives of B taken along the trajectory of a given particle in a moving medium. An application of appropriate Eulerian-Lagrangian discretization of these derivatives leads to the following semi-discrete system:

$$\begin{pmatrix} A_1 & N \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} \\ B \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} \\ g \end{pmatrix} \quad (4)$$

where

$$A_1 = \begin{pmatrix} -\mu\Delta + k^{-1} & (\mathrm{div})^* \\ \mathrm{div} & 0 \end{pmatrix}, \quad A_2 = \eta \,\mathrm{curl}\,\mathrm{curl} + k^{-1}I, \quad \text{and} \quad N(B) = (\nabla \times B) \times B.$$

This system can be discretized by using standard finite elements for the (u, p) variables in the Stokes equations and standard edge elements for the B variable; the nonlinear term $N(B)$ can be linearized by either lagging B or by using Newton's method. Further, the resulting discrete systems can then be solved very efficiently by using well-developed AMG methods: A_1^{-1} can be obtained by a preconditioned MinRes method using diagonal preconditioner given by AMG for the Poisson equation and A_2^{-1} can be approximated by applying Hiptimair-Xu preconditioners, again involving a sequence of Poisson solves. The resulting preconditioner yields a nearly optimal and highly scalable solver for these types of problems.

Several relevant technical issues and details of this approach will be discussed and reported in this presentation. For example, we will discuss how numerical quadratures that are used to evaluate the product of two finite element functions from two non-matching grids will affect the efficiency of the Eulerian-Lagrangian discretization.