Uday V. Shanbhag (UIUC) **Optimization**

Extending the Realm of Optimization for Complex Systems: Uncertainty, Competition and Dynamics
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Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ and a set $K \subseteq \mathbb{R}^n$, the variational inequality problem (VIs), denoted by VI(K, F), requires an $x \in K$ such that $(y - x)^T F(x) \geq 0$ for all $x \in K$. VIs have been employed in the context of a broad class of optimization and equilibrium problems. Yet, in settings complicated by uncertainty, the deterministic problem does not suffice; instead, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable $\xi : \Omega \to \mathbb{R}^d$ and a random map $F : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$, the stochastic variational inequality problem, denoted by SVI(K, F), requires an $x \in K$ such that

$$(y - x)^T \mathbb{E}[F(x; \xi(\omega))] \geq 0, \quad \forall y \in K, \tag{1}$$

where $\mathbb{E}[F(x; \xi)] \triangleq (\mathbb{E}[F_i(x; \xi)])_{i=1}^n$. Note that SVI$(K, F)$ can capture the solution sets of static stochastic convex optimizaton problems and Nash games in which the payoffs contain expectations. We focus on developing schemes for computing solutions of such problems in a variety of settings:

Regularized adaptive steplength stochastic approximation schemes: Stochastic approximation techniques have been an extremely popular choice of methods in contending with expectations in stochastic optimization problems. Recently, such avenues have been extended to strongly monotone stochastic variational inequalities; Recall that SVI(K, F) is strongly monotone if there exists an $\eta > 0$ such that $(y - x)^T (\mathbb{E}[F(x; \xi)] - \mathbb{E}[F(y; \xi)] \geq \eta \|x - y\|^2$ for all $x, y \in K$. In such a setting, given an $x_0 \in K$, a stochastic approximation method constructs the following sequence:

$$x_{k+1} := \Pi_K(x_k - \gamma_k F(x_k; \xi_k)), \quad k \geq 1, \tag{2}$$

where $\Pi_K(y)$ is the projection of y on K. If $\sum_{k=0}^\infty \gamma_k^2 < \infty$ and $\sum_{k=0}^\infty \gamma_k = \infty$, the $x_k \to x^*$, the unique solution of SVI(K, F), almost surely. We consider two important questions in this setting. First, can the sequence γ_k be chosen adaptively, in accordance with problem parameters (such as monotonicity constant η, Lipschitz constant L, etc.)? This is motivated by noting that traditional choices of the sequence (such as $\gamma_k = \theta/k$ require prescription of θ) and performance can vary by several orders of magnitude with choices of θ. Instead, we develop the following update rule for γ_k

$$\gamma_{k+1} = \gamma_k \left(1 - \frac{\eta \gamma_k}{2}\right), \quad k \geq 0, \tag{3}$$

based on minimizing an upper bound on error. This scheme leads to almost-sure convergence of $\{x_k\}$, adapts to problem parameters and shows good performance across multiple problems. However, such a scheme relies on the strong monotonicity of the mapping, an assumption that is often stringent, and leads us to a *regularized* scheme for monotone problems:

$$x_{k+1} := \Pi_K(x_k - \gamma_k(F(x_k; \xi_k) + \epsilon_k x_k)), \quad k \geq 1, \tag{4}$$

where $\{\epsilon_k\}$ is a decreasing sequence of positive scalars, inspired by Tikhonov regularization methods. We further extend (3) to regularized regimes where ϵ_k and γ_k are updated after every iteration.

Hybrid cutting-plane projection schemes: Next, we consider problems that arise from recourse-based stochastic equilibrium problems where feasibility is enforced in an almost-sure sense. Naturally, computing projections, while challenging, can be achieved in a scalable fashion via dual-decomposition methods for stochastic convex programs. Together with an upper-level gradient scheme, this method forms a scalable approach for solving cartesian stochastic variational inequalities. Unfortunately, in many game-theoretic settings, such an avenue is not applicable since strategy sets may be be coupled (as in the presence of network constraints). Through relaxation, we construct a scheme that operates in the primal and dual (corresponding to the shared constraints) and provide error bounds for the associated dual and inexact dual schemes (bounded complexity).