Numerical Analysis of Multiple Time Scale Dynamical Systems

Computational Analysis of Dynamical Systems

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Abstract

"Standard" algorithms for analyzing dynamical systems include initial value solvers, boundary value solvers and continuation methods. The continuation methods use root finders to track special solutions like equilibria and locate their bifurcations. Multiple time scales introduce difficulties for these methods, some of which have been studied intensively like the integration of stiff systems. This lecture will describe dynamical problems that arise in studying chemical and biological oscillations and survey our current numerical capabilities in analyzing models of dynamical systems with multiple time scales.

The Belousov-Zhabotinsky (BZ) reaction has been the most studied oscillating chemical reaction, both for its pattern forming properties in thin layers and for its complex oscillations in stirred rank reactors. Efforts that began in the 1970s to model these complex oscillations were only partially successful, in large part due to the difficulty of analyzing the models. The thesis of Chris Scheper (Cornell 2011) revisits this problem, comparing a four dimensional model proposed by Gyorgi and Field with many experimental results. Scheper's analysis of this model encounters new problems in the numerical analysis of multiple time scale systems.

Models for the electrical activity of neurons and neuronal networks take the form of nonlinear electrical circuits. Membrane channels function as nonlinear resistors whose conductance depends upon quantities such as membrane voltage, ion concentration or ligand concentration. The time scales for gating of the channels varies over several orders of magnitude: the fastest relevant time scale is about one millisecond and is associated with the basic phenomenon of action potentials. The slowest time scales are days or weeks, an example being modifications of synaptic strengths that have long been associated with memory formation. Action potentials are a key mechanism for the transmission of information in neural networks and between nerves and muscles. The complex oscillations of these systems represent their biological function directly through the frequency and timing of action potentials. We shall illustrate how advances in geometric singular perturbation theory and associated algorithms have enabled the analysis of neuronal models and describe some of the frontiers of this vibrant research area.

Slow manifolds are (locally) invariant subsets along which the speed of trajectories is commensurate with the slow time scale in slow-fast systems. In many systems, trajectories spend most of their time on attracting slow manifolds - making rapid transitions from one slow manifold to another where normal hyperbolicity of the slow manifold holds. Foundational theory for normally hyperbolic slow manifolds will be described along with numerical methods for computing them. The need for better algorithms in this area will be emphasized.

Bifurcation analysis of dynamical models maps their parameter spaces, showing which types of asymptotic behaviors are possible for different parameters. Multiple time scales give rise to new types of bifurcations, and this has long been a focus of our research. In particular, this lecture will illustrate how unstable slow manifolds and the phenomenon of canards give rise to enigmatic behaviors in simulations. The lecture will also illustrate how tangencies of invariant manifolds yield bifurcation boundaries for complex dynamical behaviors like mixed mode oscillations.