A variational problem involving a polyconvex integrand

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Abstract

Let $f \in C^1(\mathbb{R}^{d\times d})$ and $h \in C^2(0,\infty)$ be strictly convex functions. Define on $\mathbb{R}^{d\times d}$ the polyconvex function $W(\xi) = f(\xi)+h(\det\xi)$, assume that $\lim_{t\to 0^+} h(t) = \lim_{t\to\infty} h(t)/t = \infty$ and that $\mathbf{F} \in L^\infty(\Omega,\mathbb{R}^d)$ is nondegenerate. For $\mathcal{S}$ subspace of $L^\infty(\Omega,\mathbb{R}^{d\times d})$, we consider the functional

$$\mathbf{u} \to I_{\mathcal{S}}(\mathbf{u}) := \int_\Omega \Big(f(\nabla_{\mathcal{S}}\mathbf{u}) + h(\det{}^*\nabla\mathbf{u}) + \mathbf{u}\cdot\mathbf{F} \Big) dx$$

where $\nabla_{\mathcal{S}}\mathbf{u}$ is the f–projection of $\nabla\mathbf{u}$. We consider appropriately chosen finite dimensional spaces $\mathcal{S}$ such that $\nabla_{\mathcal{S}}\mathbf{u}$ converges to $\nabla\mathbf{u}$. Despite the loss of compactness of the sublevel sets of $I_{\mathcal{S}}$ and the lack of any knowledge of convexity of $I_{\mathcal{S}}$ for any metric, we show that $I_{\mathcal{S}}$ admits a unique minimizer over the set of $\mathbf{u}$ such that, in some weak sense, $\mathbf{u}(\Omega) = \Lambda$. We also uniquely characterize the minimizers by their Euler–Lagrange equations. This work is a contribution to the study of the system of PDEs $\partial_t\mathbf{u} + \mathrm{div}\Big(DW(\nabla\mathbf{u})\Big) = 0$ which remains an outstanding open problem.

This is joint work with R. Awi.