Sequences

Sequences

A sequence is an ordered collection of objects.

We use sequences to model collections in which order or multiplicity is important.

If we do not wish to impose an order, then we may choose to use bags instead.

Examples

$$\langle\rangle$$

$$\langle u, s, i, n, g, z\rangle$$

Concatenation

If s and t are sequences, we write $s \frown t$ to denote the concatenation of s and t.

$$\langle a, b, c\rangle \frown \langle d, e\rangle \ = \ \langle a, b, c, d, e\rangle$$

Filter

If s is a sequence, then $s \upharpoonright A$ is the largest subsequence of s containing only those objects that are elements of A.

$$\langle a, b, c, d, e, d, c, b, a\rangle \upharpoonright \{a, d\} \ = \ \langle a, d, d, a\rangle$$

Head and tail

If s is a non-empty sequence, then '*head* s' is the first element of s, and '*tail* s' is the remaining part.

$$head\,\langle a, b, c, d, e\rangle \ = \ a$$

$$tail\,\langle a, b, c, d, e\rangle \ = \ \langle b, c, d, e\rangle$$

Length

If s is a sequence, then we write '$\#s$' to denote the length of s.

$$\#\langle a, b, c, d, e, f\rangle = 6$$

Reverse

If s is a sequence, then we write '$reverse\ s$' to denote the sequence obtained by reversing s.

$$reverse\langle a, b, c\rangle = \langle c, b, a\rangle$$

time	from	to
10 15	OXFORD	LONDON PADDINGTON
10 38	LONDON PADDINGTON	EDINBURGH
10 40	GREAT MALVERN	LONDON PADDINGTON
11 15	MANCHESTER	POOLE
11 20	OXFORD	READING
11 40	LONDON PADDINGTON	MANCHESTER

Question

If $trains$ is

$\langle (oxford, london), (london, edinburgh), (great, london),$
$\ (manchester, poole), (oxford, reading), (london, manchester)\rangle$

what are

$$trains \upharpoonright \{\, p : Place \bullet (p, london) \,\}\ ?$$

$$tail\,(trains \upharpoonright \{\, p : Place \bullet (p, reading) \,\})\ ?$$

$$\#(trains \upharpoonright \{\, p : Place \bullet (london, p) \,\})\ ?$$

A model for sequences

A sequence may be seen as a finite function whose domain is a contiguous subset of the natural numbers.

The empty sequence may be seen as an empty function (which has an empty domain).

Any non-empty sequence has a domain which starts at 1.

Sequences as functions

If X is a set, then the set of all finite sequences of objects from X is defined by

$$\mathrm{seq}\, X == \{s : \mathbb{N} \twoheadrightarrow\!\!\!\!\!\!\;\shortmid\;\, X \mid \exists\, n : \mathbb{N} \bullet \mathrm{dom}\, s = 1 \mathinner{..} n\}$$

Concatenation

$$
\begin{array}{l}
[X] \\
\_ \frown \_ : \operatorname{seq} X \times \operatorname{seq} X \rightarrow \operatorname{seq} X \\
\hline
\forall\, s, t : \operatorname{seq} X \bullet \\
\quad \#(s \frown t) = \#s + \#t \land \\
\quad \forall\, i : 1 \mathinner{..} \#s \bullet (s \frown t)\ i \ = \ s\ i \land \\
\quad \forall\, j : 1 \mathinner{..} \#t \bullet (s \frown t)\ (j + \#s) \ = \ t\ j
\end{array}
$$

Question

How can we model *head* and *tail*?

$$
\begin{array}{l}
[X] \\
head : \\
tail : \\
\hline
\forall\, s : \operatorname{seq} X \mid \qquad \bullet \\
\quad head\ s = \\
\quad \#tail\ s = \\
\quad \forall\, i : 1 \mathinner{..} \#s - 1 \bullet (tail\ s)\ i \ =
\end{array}
$$

Recursion principle

For any constant k and function g, there is a unique total function f such that

$$
\begin{aligned}
f\ \langle\rangle &= k \\
f\ (\langle x \rangle \frown s) &= g(x, f\ s)
\end{aligned}
$$

We can use this principle to justify our definitions.

Note

The second equation is really a family of equations; there is an implicit universal quantification:

$$
\begin{array}{l}
\forall\, x : X;\ s : \operatorname{seq} X \bullet \\
\quad f\ (\langle x \rangle \frown s) = g(x, f\ s)
\end{array}
$$

Filter

$$
\begin{aligned}
\langle\rangle \upharpoonright A &= \langle\rangle && \text{(filter.1)} \\
(\langle x \rangle \frown s) \upharpoonright A &= \begin{cases} \langle x \rangle \frown (s \upharpoonright A) & \text{if } x \in A \\ s \upharpoonright A & \text{otherwise} \end{cases} && \text{(filter.2)}
\end{aligned}
$$

Reverse

$$
\begin{aligned}
reverse\ \langle\rangle &= \langle\rangle && \text{(reverse.1)} \\
reverse(\langle x \rangle \frown s) &= (reverse\ s) \frown \langle x \rangle && \text{(reverse.2)}
\end{aligned}
$$

Useful laws

$$\langle\rangle \frown t = t \qquad \text{(cat.1)}$$

$$s \frown (t \frown u) = (s \frown t) \frown u \qquad \text{(cat.2)}$$

Equational reasoning

$$\begin{aligned} & reverse(\langle\rangle \upharpoonright A) \\ &= reverse\,\langle\rangle && \text{[filter.1]} \\ &= \langle\rangle && \text{[reverse.1]} \\ &= \langle\rangle \upharpoonright A && \text{[filter.1]} \\ &= (reverse\,\langle\rangle) \upharpoonright A && \text{[reverse.1]} \end{aligned}$$

Structural induction

Some universal properties can be proved by showing that they hold of the constant $\langle\rangle$, and that they are preserved by adding any element to the front of the sequence.

Induction principle

$$\frac{P\,\langle\rangle \qquad \forall\, x : X;\ t : \mathrm{seq}\, X \bullet P\, t \Rightarrow P\,(\langle x\rangle \frown t)}{\forall\, s : \mathrm{seq}\, X \bullet P\, s}\ \text{[induction]}$$

Example

Filter is distributive:

$$\forall\, s, t : \mathrm{seq}\, X;\ A : \mathbb{P}\, X \bullet (s \frown t) \upharpoonright A = (s \upharpoonright A) \frown (t \upharpoonright A)$$

Inductive hypothesis

$$P\,\_ : \mathbb{P}\,\mathrm{seq}\, X$$

$$\forall\, s : \mathrm{seq}\, X \bullet$$
$$\quad P\, s \Leftrightarrow \forall\, t : \mathrm{seq}\, X;\ A : \mathbb{P}\, X \bullet (s \frown t) \upharpoonright A = (s \upharpoonright A) \frown (t \upharpoonright A)$$

Proof outline

$$\dfrac{\dfrac{P\,\langle\rangle}{} \text{[Lemma 1]} \qquad \dfrac{\dfrac{\dfrac{[x \in X \land r \in \operatorname{seq} X]^{[1]} \quad [P\,r]^{[2]}}{P\,(\langle x\rangle \frown r)}\text{[Lemma 2]}}{P\,r \Rightarrow P\,(\langle x\rangle \frown r)}[\Rightarrow\!-\text{intro}^{[2]}]}{\forall\, x : X;\ r : \operatorname{seq} X \bullet P\,r \Rightarrow P\,(\langle x\rangle \frown r)}[\forall\!-\text{intro}^{[1]}]}{\dfrac{\forall\, s : \operatorname{seq} X \bullet P\,s}{\forall\, s, t : \operatorname{seq} X;\ A : \mathbb{P}\,X \bullet (s \frown t) \upharpoonright A = (s \upharpoonright A) \frown (t \upharpoonright A)}\text{[axdef]}}\text{[induction]}$$

Lemma 1

$$\begin{aligned} & (\langle\rangle \frown t) \upharpoonright A \\ &= t \upharpoonright A && \text{[cat.1]} \\ &= \langle\rangle \frown (t \upharpoonright A) && \text{[cat.1]} \\ &= (\langle\rangle \upharpoonright A) \frown (t \upharpoonright A) && \text{[filter.1]} \end{aligned}$$

Lemma 2

$$\begin{aligned} & ((\langle x\rangle \frown r) \frown t) \upharpoonright A \\ &= (\langle x\rangle \frown (r \frown t)) \upharpoonright A && \text{[cat.2]} \\ &= \begin{cases} \langle x\rangle \frown ((r \frown t) \upharpoonright A) & \text{if } x \in A \\ (r \frown t) \upharpoonright A & \text{otherwise} \end{cases} && \text{[filter.2]} \\ &= \begin{cases} \langle x\rangle \frown ((r \upharpoonright A) \frown (t \upharpoonright A)) & \text{if } x \in A \\ (r \upharpoonright A) \frown (t \upharpoonright A) & \text{otherwise} \end{cases} && [P\,r] \\ &= \begin{cases} (\langle x\rangle \frown (r \upharpoonright A)) \frown (t \upharpoonright A) & \text{if } x \in A \\ (r \upharpoonright A) \frown (t \upharpoonright A) & \text{otherwise} \end{cases} && \text{[cat.2]} \\ &= ((\langle x\rangle \frown r) \upharpoonright A) \frown (t \upharpoonright A) && \text{[filter.2]} \end{aligned}$$

Presentation

- inductive property
- base case
- inductive step

Bags

A bag is an unordered collection of objects in which multiplicities are important:

$[\![\ ]\!]$

$[\![a, b, c, a, b, c]\!]$

A model for bags

If X is a set, then the set of all bags of elements from X is defined by

$$\operatorname{bag} X \ == \ X \nrightarrow \mathbb{N} \setminus \{0\}$$

Summary

- ordered collections
- $\frown$, *head*, *tail*, $\upharpoonright$, and *reverse*
- sequences as functions
- algebraic laws
- structural induction
- bags