Relations

Relations

We use sets of pairs called relations to model relationships between objects.

The presence of a pair of objects in the set indicates that the first is related to the second.

Notation

If X and Y are sets, then $X \leftrightarrow Y$ denotes the set of all relations between X and Y:

$$X \leftrightarrow Y \;==\; \mathbb{P}(X \times Y)$$

The pair (p, q) can also be written as $p \mapsto q$.

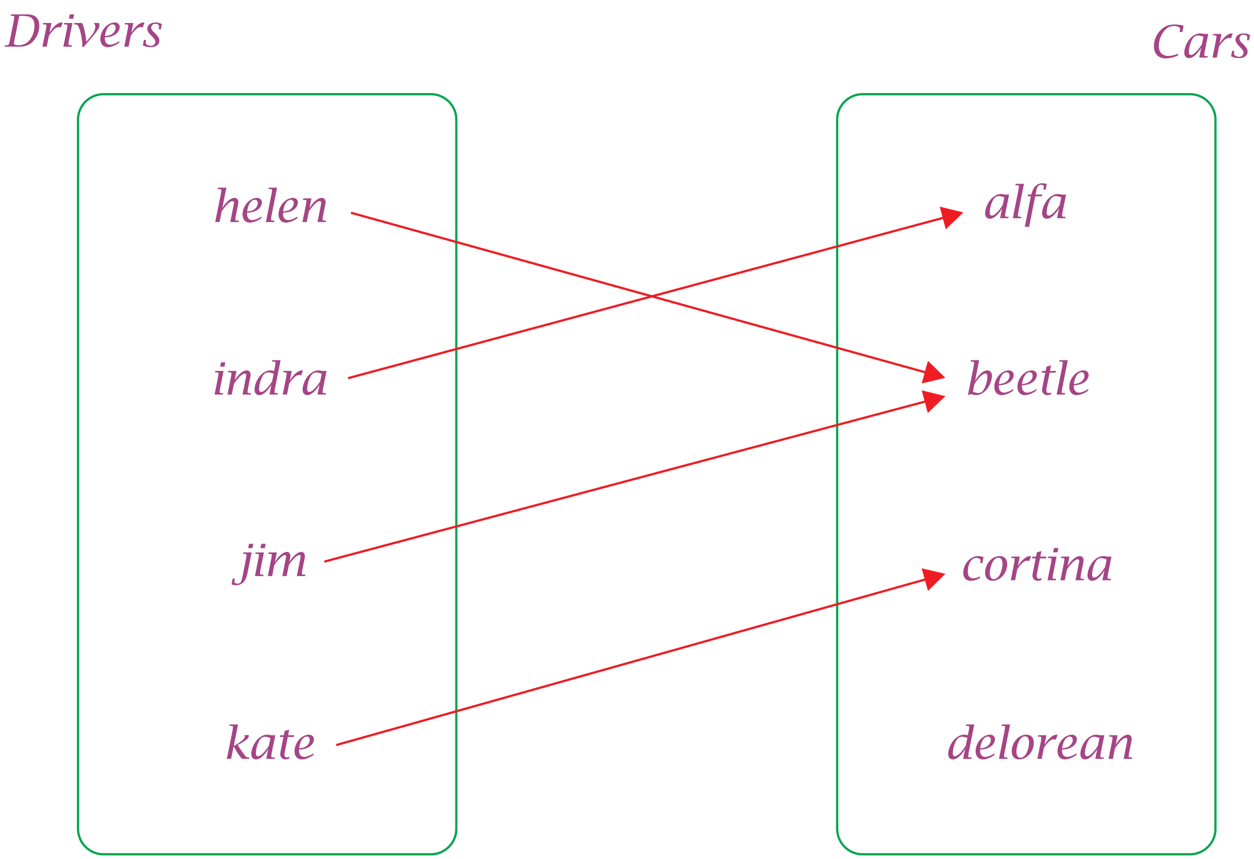
Example

$$
\begin{aligned}
\{a, b\} \leftrightarrow \{0, 1\} \;=\; \{&\emptyset, \\
&\{(a,0)\}, \{(a,1)\}, \{(b,0)\}, \{(b,1)\}, \\
&\{(a,0),(a,1)\}, \{(a,0),(b,0)\}, \\
&\{(a,0),(b,1)\}, \{(a,1),(b,0)\}, \\
&\{(a,1),(b,1)\}, \{(b,0),(b,1)\}, \\
&\{(a,0),(a,1),(b,0)\}, \\
&\{(a,0),(a,1),(b,1)\}, \\
&\{(a,0),(b,0),(b,1)\}, \\
&\{(a,1),(b,0),(b,1)\}, \\
&\{(a,0),(a,1),(b,0),(b,1)\}\}
\end{aligned}
$$

Example

$$\begin{aligned} \mathit{Drivers} &== \{\mathit{helen}, \mathit{indra}, \mathit{jim}, \mathit{kate}\} \\ \mathit{Cars} &== \{\mathit{alfa}, \mathit{beetle}, \mathit{cortina}, \mathit{delorean}\} \end{aligned}$$

$$\begin{aligned} \mathit{drives} == \{&\mathit{helen} \mapsto \mathit{beetle}, \mathit{indra} \mapsto \mathit{alfa}, \\ &\mathit{jim} \mapsto \mathit{beetle}, \mathit{kate} \mapsto \mathit{cortina}\} \end{aligned}$$


Drivers
Cars
helen
indra
jim
kate
alfa
beetle
cortina
delorean

Domain and range

domain:

$$\operatorname{dom} R \;=\; \{\, x : X;\; y : Y \mid x \mapsto y \in R \bullet x \,\}$$

range:

$$\operatorname{ran} R \;=\; \{\, x : X;\; y : Y \mid x \mapsto y \in R \bullet y \,\}$$

Example

$$\mathrm{dom}\ drives = \{helen, indra, jim, kate\}$$

$$\mathrm{ran}\ drives = \{alfa, beetle, cortina\}$$

Restrictions

domain restriction:

$$A \lhd R \ = \ \{\, x : X;\ y : Y \mid x \mapsto y \in R \land x \in A \bullet x \mapsto y \,\}$$

range restriction:

$$R \rhd B \ = \ \{\, x : X;\ y : Y \mid x \mapsto y \in R \land y \in B \bullet x \mapsto y \,\}$$

Subtraction

domain subtraction:

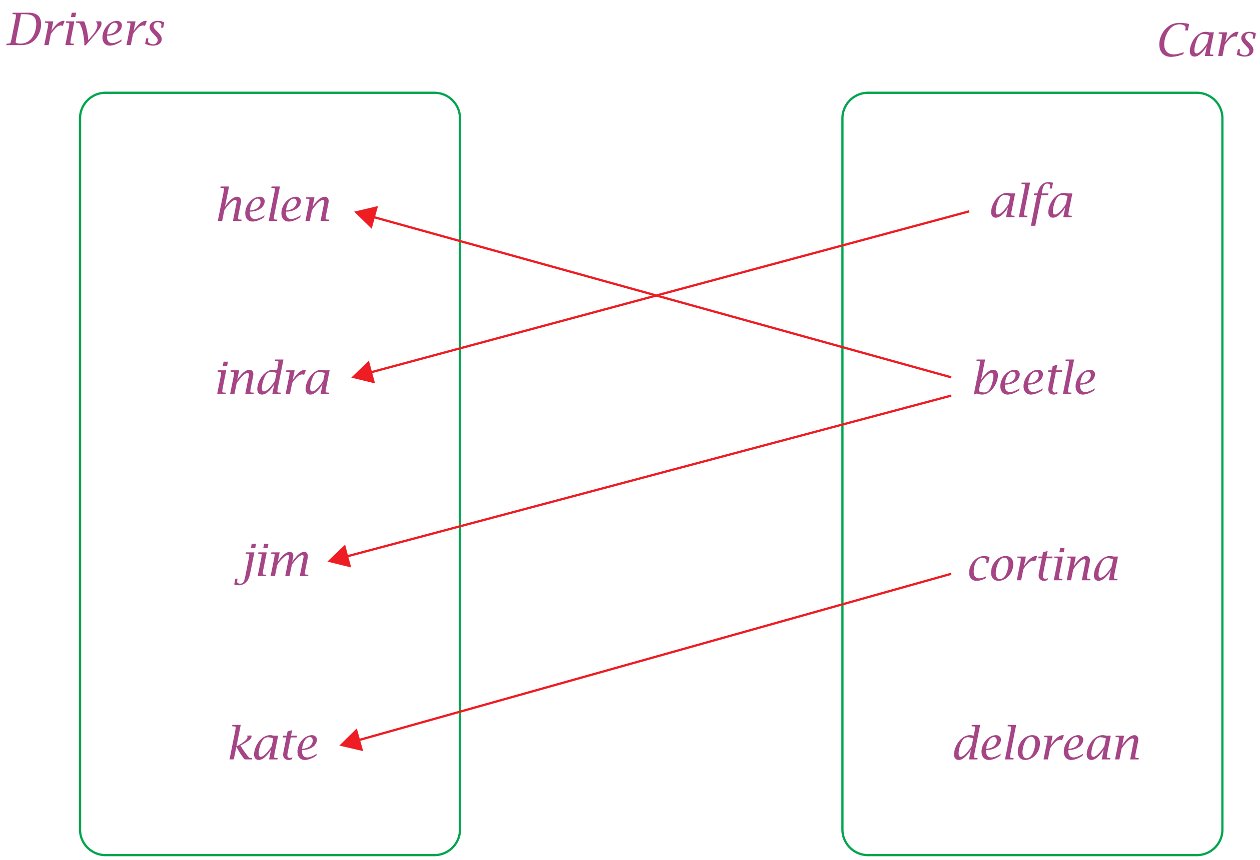
$$A \mathbin{\lhd\!\!\!-} R \;=\; \{\, x : X;\ y : Y \mid x \mapsto y \in R \land x \notin A \bullet x \mapsto y \,\}$$

range subtraction:

$$R \mathbin{\rhd\!\!\!-} B \;=\; \{\, x : X;\ y : Y \mid x \mapsto y \in R \land y \notin B \bullet x \mapsto y \,\}$$

Inverse

$$\forall\, x : X;\ y : Y \bullet x \mapsto y \in R^{\sim} \;\Rightarrow\; y \mapsto x \in R$$

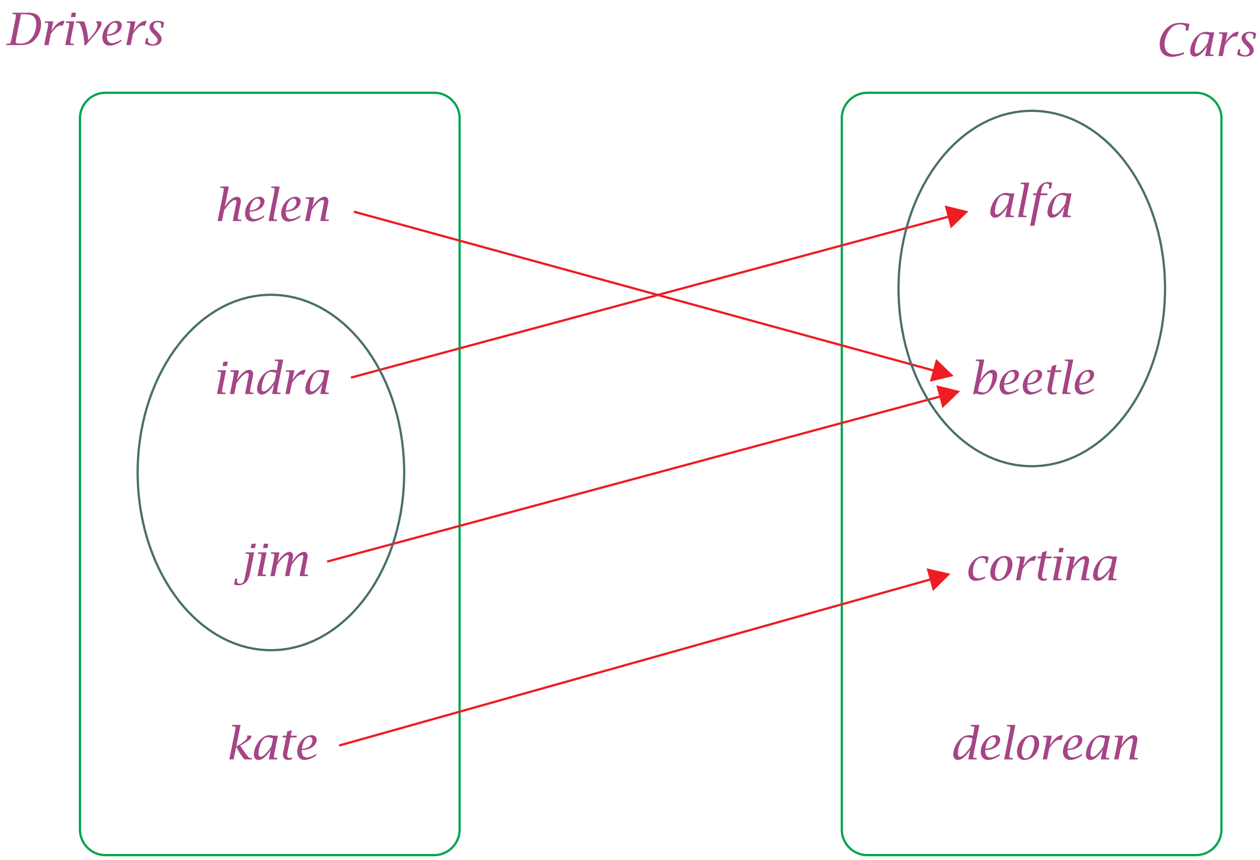

Drivers
Cars
helen
indra
jim
kate
alfa
beetle
cortina
delorean

Relational image

$$R(\!|\, A \,|\!) \;=\; \operatorname{ran}(A \lhd R)$$

Example

$$drives(\!|\ \{indra, jim\}\ |\!) = \{alfa, beetle\}$$


Drivers
Cars
helen
indra
jim
kate
alfa
beetle
cortina
delorean

Question

What do the following sets look like in extension?

- $\{jim, kate\} \lhd drives$
- $drives \rhd \{beetle\}$
- $\mathrm{dom}(drives^{\sim})$
- $drives^{\sim}(\!| \{beetle, delorean\} |\!)$

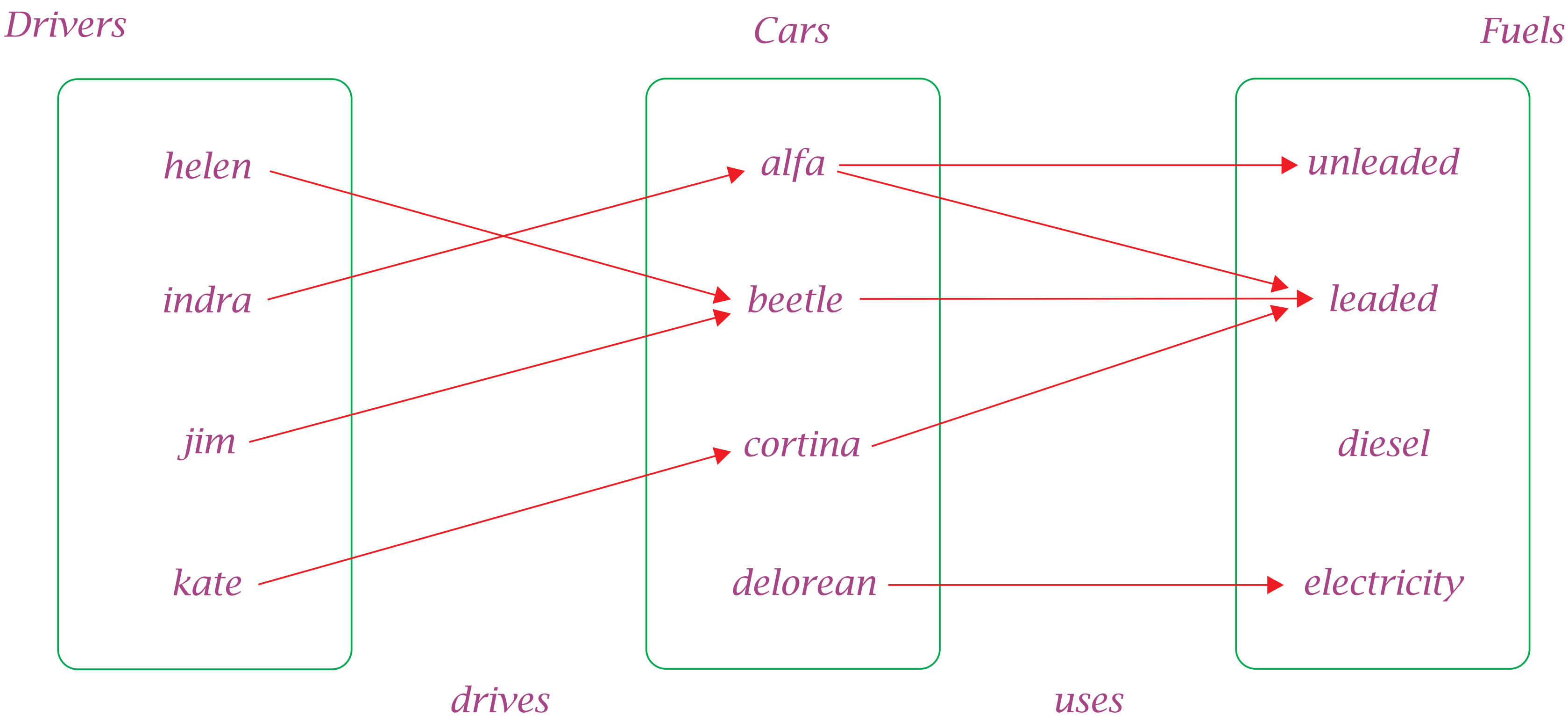
Relational composition

If the target of one relation matches the source of another, it may be useful to consider their relational composition.

The composition of two relations relates objects in the source of the first to objects in the target of the second, provided that some intermediate point exists.

Definition

$$x \mapsto z \in R \mathbin{;} S \iff \exists y : Y \bullet x \mapsto y \in R \land y \mapsto z \in S$$



Drivers
Cars
Fuels
helen
indra
jim
kate
alfa
beetle
cortina
delorean
unleaded
leaded
diesel
electricity
drives
uses

Question

Which of the following statements are true?

- $\mathit{jim} \mapsto \mathit{unleaded} \in (\mathit{drives} \comp \mathit{uses})$
- $\mathit{electricity} \in \mathrm{ran}(\mathit{drives} \comp \mathit{uses})$
- $\mathit{indra} \mapsto \mathit{unleaded} \in (\mathit{drives} \comp \mathit{uses})$

Properties

- reflexivity
- symmetry (and …)
- transitivity

Reflexivity and symmetry

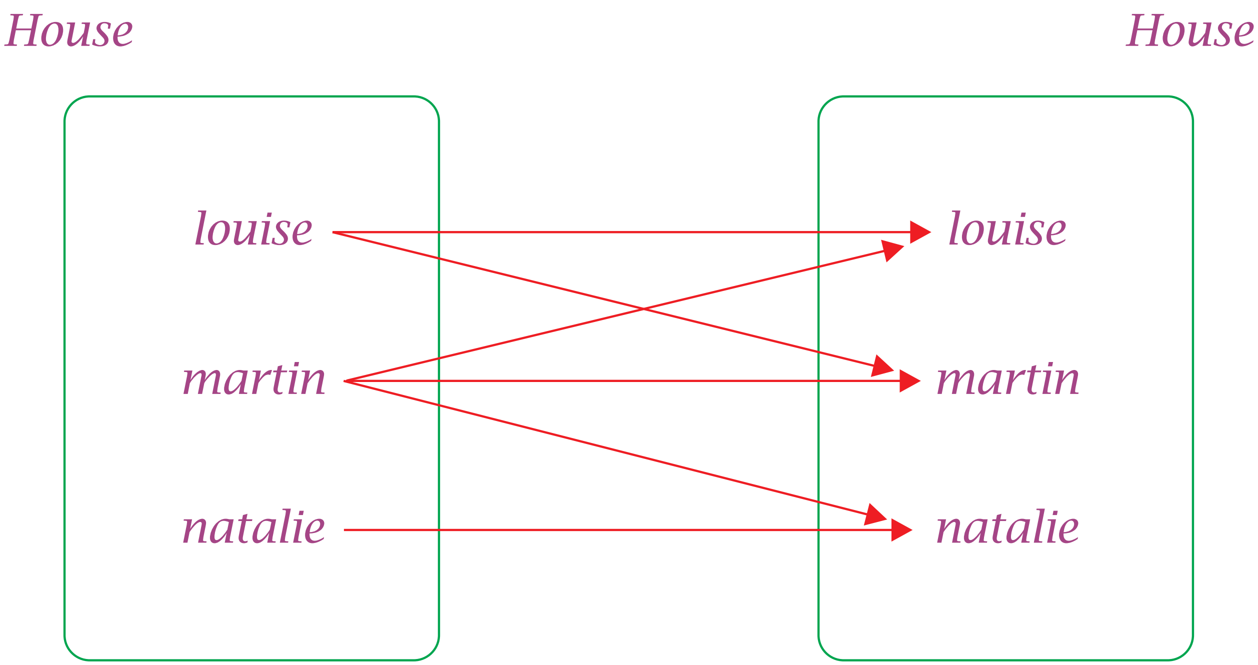
$$
\begin{aligned}
\mathit{Reflexive}[X] == \{\, R : X \leftrightarrow X \mid & \\
& \forall\, x : X \bullet \\
& \quad x \mapsto x \in R \,\}
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Symmetric}[X] == \{\, R : X \leftrightarrow X \mid & \\
& \forall\, x, y : X \bullet \\
& \quad x \mapsto y \in R \Rightarrow y \mapsto x \in R \,\}
\end{aligned}
$$

Transitivity

$$
\begin{aligned}
\mathit{Transitive}[X] == \{\, R : X \leftrightarrow X \mid \\
\forall\, x, y, z : X \bullet \\
x \mapsto y \in R \land y \mapsto z \in R \Rightarrow \\
x \mapsto z \in R \,\}
\end{aligned}
$$

$$
\begin{aligned}
\mathit{Equivalence}[X] == \mathit{Reflexive}[X] \cap \\
\mathit{Symmetric}[X] \cap \\
\mathit{Transitive}[X]
\end{aligned}
$$


House
House
louise
martin
natalie
louise
martin
natalie

Closure

If a relation does not have a specified property, then we may add maplets until it does.

If we add precisely those maplets that are lacking, then the resulting relation is called a closure.

Reflexive closure

To obtain the reflexive closure R^{r} of a relation R, we have only to add the maplets of the identity relation:

$$\mathrm{id}\, X \;==\; \{\, x : X \bullet x \mapsto x \,\}$$

Question

What should we add in order to obtain the symmetric closure of a relation S?

Iteration

$$
\begin{aligned}
R^1 &= R \\
R^2 &= R \fatsemi R \\
R^3 &= R \fatsemi R \fatsemi R \\
&\vdots
\end{aligned}
$$

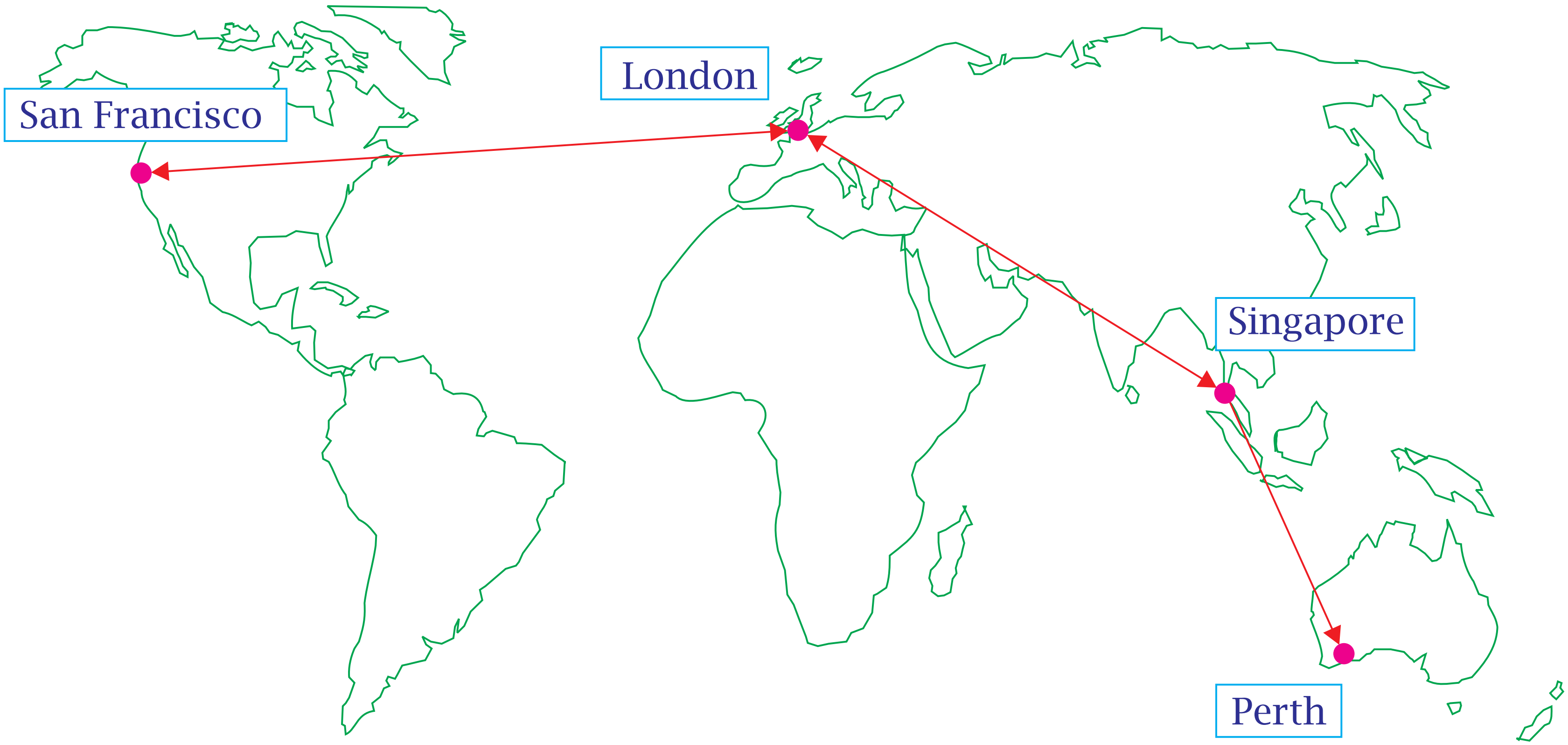
Transitive closures

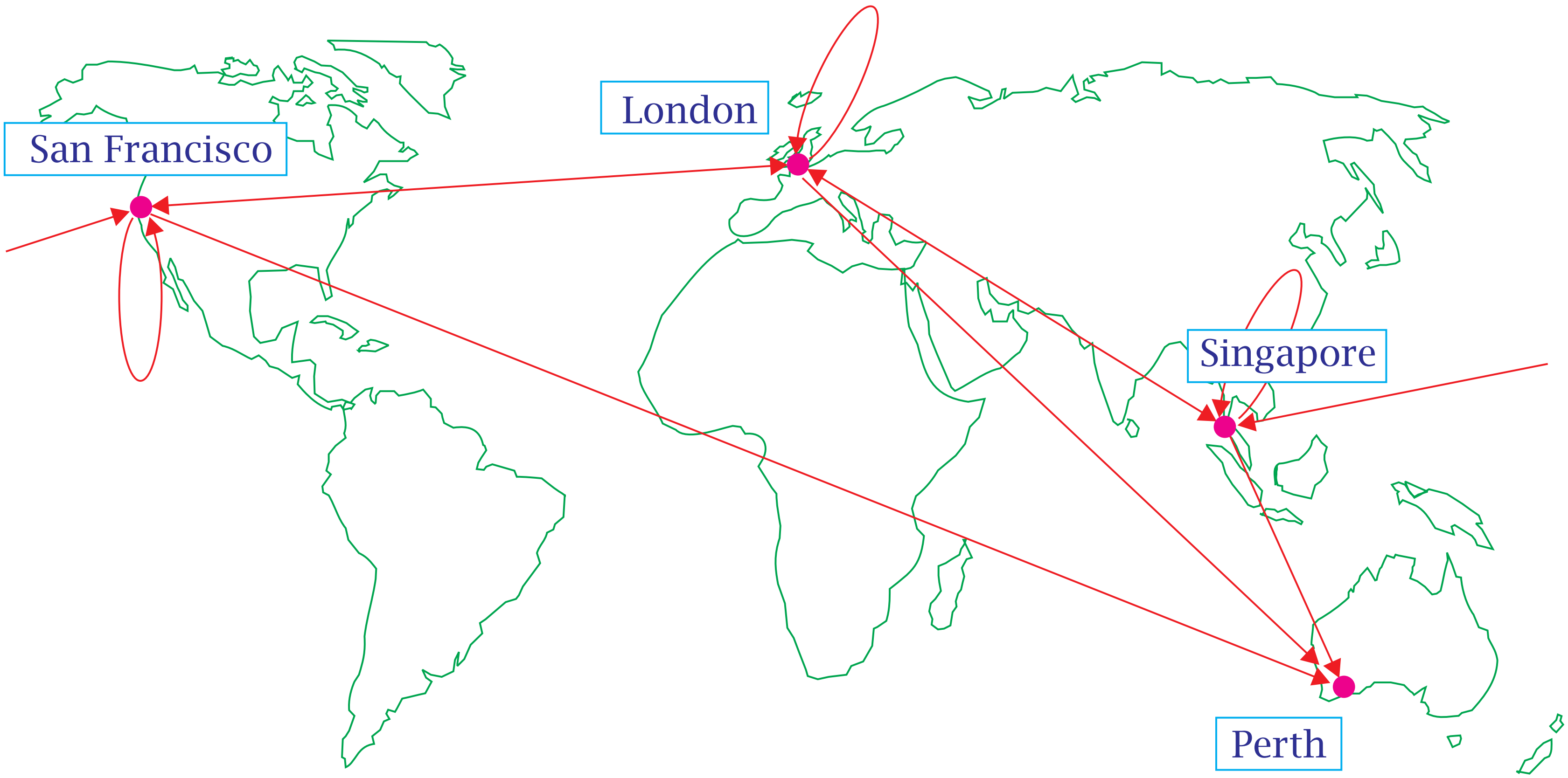
$$R^{+} = \bigcup \{ n : \mathbb{N} \mid n \geq 1 \bullet R^{n} \}$$

$$R^{*} = R^{+} \cup \mathrm{id}\, X$$

Example

$$\begin{aligned} direct \;=\; \{ & singapore \mapsto london, london \mapsto singapore, \\ & singapore \mapsto perth, london \mapsto san\_francisco, \\ & san\_francisco \mapsto london\} \end{aligned}$$


London
San Francisco
Singapore
Perth


London
San Francisco
Singapore
Perth

Proof strategy

If $\preceq$ is transitive, then the following rule is valid:

$$\frac{a \preceq b \quad b \preceq c}{a \preceq c}\ [\preceq \text{ is transitive}]$$

Presentation

We write

$$\dfrac{\dfrac{\overline{a \preceq b}\ \text{[reason 1]} \qquad \overline{b \preceq c}\ \text{[reason 2]}}{a \preceq c} \qquad \overline{c \preceq d}\ \text{[reason 3]}}{a \preceq d}$$

as

$$\begin{array}{ll} a & \\ \preceq b & \text{[reason 1]} \\ \preceq c & \text{[reason 2]} \\ \preceq d & \text{[reason 3]} \end{array}$$

Summary

- relationships between objects
- $\leftrightarrow$, $\mapsto$
- dom, ran, $\lhd$, $\rhd$, $⩤$, $⩥$, $(\!|\ \ |\!)$
- $\fatsemi$, $\sim$
- reflexivity, symmetry, transitivity
- R^{r}, S^{s}, T^{+}, T^{*}
- transitive relations