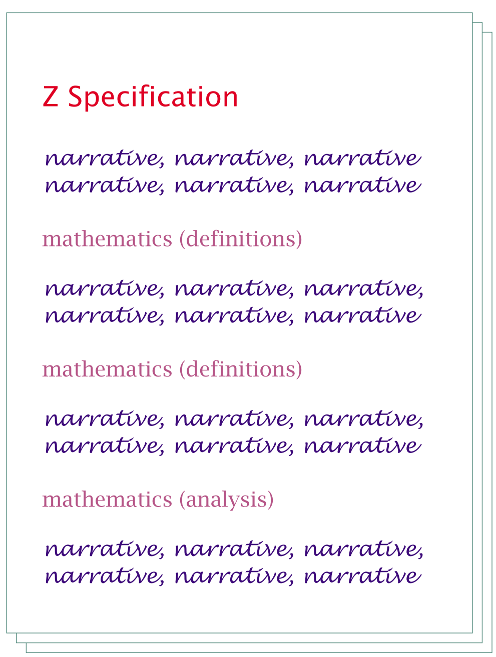
Definitions

Document



Definitions

- declaration
- abbreviation
- axiom
- free types
- schemas

Basic type declarations

We may introduce the name for a new basic type simply by writing it between a pair of brackets:

$$[\mathit{Type}]$$

Once this has been done, we may introduce variables as elements of this type.

Abbreviations

An abbreviation introduces a new name x for an object e that has been already defined.

$$x \;==\; e$$

Following this definition, we may infer that

$$\frac{}{x = e}\;[\text{abbreviation}]$$

Example

$$\mathit{Addictive} \;==\; \{\mathit{red}, \mathit{green}, \mathit{blue}\}$$

Example

$$\begin{aligned} n! &== n * (n-1)! \\ 0! &== 1 \end{aligned}$$

Axiomatic definitions

An axiomatic definition introduces a new global constant under a constraint:

$$\begin{array}{|l} x : S \\ \hline p \end{array}$$

Following this definition, we may infer that

$$\frac{}{x \in S \land p}\ [\text{axiom}]$$

Example

$$\begin{array}{|l} \mathit{maxsize} : \mathbb{N} \\ \hline \mathit{maxsize} > 0 \end{array}$$

Consistency

A definition is consistent if it does not contradict any of the other statements in the document.

To show that a definition is consistent, we have only to show that an object exists with the specified property.

To show that the axiomatic definition

$$\begin{array}{|l} x : S \\ \hline p \end{array}$$

is consistent, it is enough to show that

$$\exists\, x : S \bullet p$$

Example

The following definition is not consistent:

$$\begin{array}{|l} \mathit{maxprime} : \mathbb{N} \\ \hline \mathit{maxprime} \in \mathit{Primes} \\ \forall\, p : \mathit{Primes} \bullet \mathit{maxprime} \geq p \end{array}$$

Question

What if there is no specified property? Can the following introduce a contradiction?

$$x : S$$

Generic definitions

Some objects are generic; there may be different instances of the same object for different sets or types.

A generic object may be defined using one or more generic parameters, which may be enclosed in square brackets.

If the values of the parameters are obvious from the context in which the object appears, we may choose to omit them.

Generic abbreviations

A generic abbreviation introduces a family of symbols, indexed by one or more set parameters:

$$x\,p \;==\; e$$

Following this definition, we may infer that

$$\frac{}{x\,q = e[q/p]}\;[\text{abbreviation}]$$

Example

Given the abbreviation

$$\varnothing\,[S] \;==\; \{\, x : S \mid \mathit{false} \,\}$$

we may infer that

$$\varnothing\,[\mathbb{N}] = \{\, x : \mathbb{N} \mid \mathit{false} \,\}$$

Generic axiomatic definitions

A generic axiomatic definition introduces a family of symbols with specified properties:

$$[X]$$
$$x : S$$
$$p$$

Example

We could have defined the empty set using a generic axiomatic definition instead of a generic abbreviation:

$$[X]$$
$$\varnothing : \mathbb{P}\, X$$
$$\forall\, x : X \bullet x \notin \varnothing$$

The same effect could have been achieved by providing a separate axiomatic definition for each instantiation of $\varnothing$:

$$\varnothing[\mathit{Car}] : \mathbb{P}\,\mathit{Car}$$
$$\forall\, x : \mathit{Car} \bullet x \notin \varnothing[\mathit{Car}]$$

$$\varnothing[\mathit{Person}] : \mathbb{P}\,\mathit{Person}$$
$$\forall\, x : \mathit{Person} \bullet x \notin \varnothing[\mathit{Person}]$$

These axiomatic definitions justify the following:

$$\overline{\varnothing[\mathit{Car}] \in \mathbb{P}\,\mathit{Car} \wedge \forall\, x : \mathit{Car} \bullet x \notin \varnothing[\mathit{Car}]}$$

and

$$\overline{\varnothing[\mathit{Person}] \in \mathbb{P}\,\mathit{Person} \wedge \forall\, x : \mathit{Person} \bullet x \notin \varnothing[\mathit{Person}]}$$

Information

After the generic definition

$$\begin{array}{|l}
\hline\hline [X] \\
x : S \\
\hline
p \\
\hline
\end{array}$$

we may infer

$$\frac{}{(x \in S \land p)[T / X][x[T] / x]} \text{ [generic axiom]}$$

Example

For any set T, we have that

$$\frac{}{\varnothing[T] \in \mathbb{P}\, T \land \forall x : T \bullet x \notin \varnothing[T]}$$

Example

$$[X]$$

$$\_ \subseteq \_ : \mathbb{P}\,X \leftrightarrow \mathbb{P}\,X$$

$$\forall\, s, t : \mathbb{P}\,X \bullet$$
$$\quad s \subseteq t \Leftrightarrow \forall\, x : X \bullet x \in s \Rightarrow x \in t$$

$$\{2, 3\} \subseteq \{1, 2, 3, 4\}$$

Characteristic sets

In reasoning about a property, it is often convenient to identify the property with the set of all objects that possess it: the characteristic set of property p in type t is given by

$$c = \{ x : t \mid p \}$$

Example

$$\begin{array}{|l} crowds : \mathbb{P}(\mathbb{P}\, Person) \\ \hline crowds = \{\, s : \mathbb{P}\, Person \mid \#s \geq 3 \,\} \end{array}$$

$\{Alice, Bill, Claire\} \in crowds$

$\{Dave, Edward\} \notin crowds$

Example

$$\begin{array}{|l} safe\_ : \mathbb{P}(\mathbb{P}\, Person) \\ \hline \forall\, s : \mathbb{P}\, Person \bullet safe\ s \Leftrightarrow \neg(\{Alice, Bill\} \subseteq s) \end{array}$$

$safe\ \{Alice, Claire, Dave\}$

$\neg\ (safe\ \{Alice, Bill, Edward\})$

Summary

- basic type declarations
- abbreviations
- axiomatic definitions
- contradictions
- generic definitions
- characteristic sets