Sets

Sets

- a set is any well-defined collection of objects
- small sets may be introduced by listing their elements
- other sets may be constructed using set comprehension, the power set operator, and the Cartesian product.

Examples

- the four oceans of the world
- the individuals who have been appointed to the post of secretary-general of the United Nations
- the prime numbers
- the collection of programs written in C^{++} that halt if run for a sufficient time on a computer with unlimited storage

Extension

- $\mathit{Oceans} \;==\; \{\mathit{Atlantic},\ \mathit{Arctic},\ \mathit{Indian},\ \mathit{Pacific}\}$
- $\mathit{Secretaries\text{-}General} \;==$

$$\{\ \mathit{Trygve\ Lie},\ \mathit{Dag\ Hammarskjöld},\ \mathit{U\ Thant},\ \mathit{Kurt\ Waldheim},\ \mathit{Javier\ Pérez\ de\ Cuéllar},\ \mathit{Boutros\ Boutros\ Ghali},\ \mathit{Kofi\ Annan}\ \}$$

Membership

We write $x \in s$ to indicate that x is an element of set s.

We write $\neg(x \in s)$ as $x \notin s$.

Example

- $3 \in \mathit{Primes}$
- $5 \in \mathit{Primes}$
- $8 \notin \mathit{Primes}$

Membership and extension

An object t is an element of a set written in extension iff it is equal to one of the expressions within the braces:

$$\frac{t = u_1 \lor \ldots \lor t = u_n}{t \in \{ u_1, \ldots, u_n \}} \text{ [membership and extension]}$$

Extensionality

Two sets are equal iff every element of one is also an element of the other:

$$\frac{(\forall\, x : t \bullet x \in u) \land (\forall\, x : u \bullet x \in t)}{t = u}\ [\text{extensionality}]$$

provided that x is free in neither u nor t

Subset

A set s is a subset of another set t iff every element of s is also an element of t:

$$\frac{\forall x : s \bullet x \in t}{s \subseteq t} \; [\text{subset}]$$

provided that x is not free in t

Examples

- $\{1, 2, 3\} \subseteq \{0, 1, 2, 3, 4\}$
- $\{4, 6, 8\} \subseteq \{4, 6, 8\}$

The empty set

$$\frac{}{\forall\, x : a \bullet x \notin \emptyset}\ [\text{empty set}]$$

The natural numbers

$$\mathbb{N} \;=\; \{0, 1, 2, 3, 4, 5, \ldots\}$$

Set comprehension

We may restrict our attention to those elements of a set s that possess a property p:

$$\frac{t \in \{ x : s \mid p \}}{t \in s \land p[t/x]} \text{ [comprehension]}$$

Example

$$jim \in \{\, p : Person \mid p \text{ eats cheese} \,\}$$

Term comprehension

We may also describe a set of objects constructed from certain elements of a given set:

$$\frac{f \in \{ x : s \mid p \bullet e \}}{\exists\, x : s \mid p \bullet e = f} \; [\text{term comprehension}]$$

provided that x is not free in f x

Example

$$56, \text{West Street} \in \{\, p : \mathit{Person} \mid p \text{ eats cheese} \bullet \mathit{address}(p) \,\}$$

Power set

$$\frac{s \subseteq a}{s \in \mathbb{P}\, a} \; [\text{power}]$$

Examples

- $\mathbb{P}\{x, y\} \;=\; \{\,\varnothing, \{x\}, \{y\}, \{x, y\}\,\}$
- $\{1, 2, 3, 4\} \in \mathbb{P}\,\mathbb{N}$
- $\mathbb{P}\,\varnothing \;=\; \ldots?$

Cartesian product

If a and b are sets, then the Cartesian product $a \times b$ consists of all tuples of the form (x, y), where x is an element of a and y is an element of b.

$$\frac{x_1 \in a_1 \wedge \ldots \wedge x_n \in a_n}{(x_1, \ldots, x_n) \in a_1 \times \ldots \times a_n} \text{ [tuple membership]}$$

Example

$$
\begin{aligned}
\mathit{Days} &= \{\, 1, 2, \ldots, 31 \,\} \\
\mathit{Months} &= \{\, \mathit{January}, \mathit{February}, \ldots, \mathit{December} \,\} \\
\mathit{Years} &= \{\, 1900, 1901, \ldots, 1999 \,\}
\end{aligned}
$$

$$(22, \mathit{November}, 1990) \;\in\; \mathit{Days} \times \mathit{Months} \times \mathit{Years}$$

Ordered tuples

Two tuples are equal if they have exactly the same elements, and these are arranged in the same order:

$$\frac{x_1 = y_1 \wedge \ldots \wedge x_n = y_n}{(x_1, \ldots, x_n) = (y_1, \ldots, y_n)} \text{ [tuple equality]}$$

Component selection

We may select a component of an ordered tuple using a simple dot notation: if m is an integer between 1 and n, then

$$(x_1, x_2, \ldots, x_n) \, . \, m \;=\; x_m$$

Example

$$date = (22, November, 1990)$$

$$\begin{aligned} date.1 &= 22 \\ date.2 &= November \\ date.3 &= 1990 \end{aligned}$$

Questions

$$\{\, x : \mathit{Person};\ y : \mathit{Person} \mid x \text{ is associated with } y \land$$
$$x \text{ eats cheese} \land$$
$$y \text{ reads Hello} \bullet x \,\}$$

- what if we are looking for both of them?
- what if we omit the term part?

Characteristic tuple

The expression

$$\{\ a : \mathbb{N};\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \land a \notin t\ \}$$

has the same value as

$$\{\ a : \mathbb{N};\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \land a \notin t \bullet (a, s, t)\ \}$$

Question

Can you exhibit an element of each of the following sets?

- $\{ a : \mathbb{N};\ b : \mathbb{N} \mid a > b \}$
- $\{ s : \mathbb{P}\,\mathbb{N};\ a : \mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \wedge a \in t \}$

Union

$$\frac{x \in (a \cup b)}{x \in a \vee x \in b} \text{ [union]}$$

Example

$$\begin{aligned} \mathit{Guests} \;=\; & \text{Jim's friends} \\ & \cup \\ & \text{Eleanor's friends} \end{aligned}$$

Intersection

$$\frac{x \in (a \cap b)}{x \in a \wedge x \in b} \text{ [intersection]}$$

Example

$$\textit{Steve} \in \textit{Choir} \cap \textit{1st VIII} \cap \textit{1st XV}$$

Difference

$$\frac{x \in (a \setminus b)}{x \in a \wedge x \notin b}\ [\text{diff}]$$

Example

$$Jim \in Voters \setminus Conservatives$$

Distributed forms

$$\frac{x \in \bigcup s}{\exists\, a : s \bullet x \in a} \quad \text{[distributed union]}$$

$$\frac{x \in \bigcap s}{\forall\, a : s \bullet x \in a} \quad \text{[distributed intersection]}$$

Example

$$\mathit{Europeans} = \bigcup\{\mathit{Belgians}, \mathit{British}, \mathit{Danes}, \mathit{Dutch}, \mathit{French}, \mathit{Germans}, \mathit{Greeks}, \mathit{Irish}, \mathit{Italians}, \mathit{Luxemburgers}, \mathit{Portuguese}, \mathit{Spanish}, \mathit{Swedes}, \ldots\}$$

$$\{\mathit{Steve}\} = \bigcap\{\mathit{Choir}, \mathit{1st\ VIII}, \mathit{1st\ XV}\}$$

Types

A type is a maximal set, at least within the confines of the current specification.

This ensures that each object x is associated with exactly one type: the largest set s present for which $x \in s$.

Integers

There is a single built in type, $\mathbb{Z}$, the type of all integers

$$\mathbb{Z} \;=\; \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Other types may be constructed from $\mathbb{Z}$, or from user-defined basic types.

Examples

- $\mathbb{Z} \times \mathbb{Z}$
- $\mathbb{P}\,\mathbb{Z}$
- *People*
- $\ldots$

Advantages

- adds structure and readability
- correctness via type-checking
- data types suggest implementation

Question

What is the type of the empty set?

Summary

- collections of objects
- membership
- extensionality
- comprehension
- $\mathbb{P}$ and $\times$
- types