Sets

Sets

- a set is any well-defined collection of objects
- small sets may be introduced by listing their elements
- other sets may be constructed using set comprehension, the power set operator, and the Cartesian product.

Examples

- the four oceans of the world
- the individuals who have been appointed to the post of secretary-general of the United Nations
- the prime numbers
- the collection of programs written in C^{++} that halt if run for a sufficient time on a computer with unlimited storage

Extension

- $\mathit{Oceans} == \{\mathit{Atlantic}, \mathit{Arctic}, \mathit{Indian}, \mathit{Pacific}\}$
- *Secretaries-General* ==
  { *Trygve Lie*, *Dag Hammarskjöld*, *U Thant*,
  Kurt Waldheim, *Javier Pérez de Cuéllar*,
  Boutros Boutros Ghali, *Kofi Annan* }

Membership

We write $x \in s$ to indicate that x is an element of set s.

We write $\neg(x \in s)$ as $x \notin s$.

Example

- $3 \in \mathit{Primes}$
- $5 \in \mathit{Primes}$
- $8 \notin \mathit{Primes}$

Membership and extension

An object t is an element of a set written in extension iff it is equal to one of the expressions within the braces:

$$\frac{t = u_1 \lor \ldots \lor t = u_n}{t \in \{ u_1, \ldots, u_n \}} \text{ [membership and extension]}$$

Extensionality

Two sets are equal iff every element of one is also an element of the other:

$$\frac{(\forall x : t \bullet x \in u) \land (\forall x : u \bullet x \in t)}{t = u} \text{ [extensionality]}$$

provided that x is free in neither u nor t

Subset

A set s is a subset of another set t iff every element of s is also an element of t:

$$\frac{\forall x : s \bullet x \in t}{s \subseteq t} \text{ [subset]}$$

provided that x is not free in t

Examples

- $\{1, 2, 3\} \subseteq \{0, 1, 2, 3, 4\}$
- $\{4, 6, 8\} \subseteq \{4, 6, 8\}$

The empty set

$$\frac{}{\forall x : a \bullet x \notin \varnothing} \text{ [empty set]}$$

The natural numbers

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

Set comprehension

We may restrict our attention to those elements of a set s that possess a property p:

$$\frac{t \in \{ x : s \mid p \}}{t \in s \land p[t/x]} \; [\text{comprehension}]$$

Example

$$jim \in \{ p : Person \mid p \text{ eats cheese} \}$$

Term comprehension

We may also describe a set of objects constructed from certain elements of a given set:

$$\frac{f \in \{ x : s \mid p \bullet e \}}{\exists x : s \mid p \bullet e = f} \; [\text{term comprehension}]$$

provided that x is not free in fX

Example

$$56, \text{West Street} \in \{ p : Person \mid p \text{ eats cheese} \bullet address(p) \}$$

Power set

$$\frac{s \subseteq a}{s \in \mathbb{P}\, a} \; [\text{power}]$$

Examples

- $\mathbb{P}\{x, y\} = \{ \emptyset, \{x\}, \{y\}, \{x, y\} \}$
- $\{1, 2, 3, 4\} \in \mathbb{P}\, \mathbb{N}$
- $\mathbb{P}\, \emptyset = \ldots ?$

Cartesian product

If a and b are sets, then the Cartesian product $a \times b$ consists of all tuples of the form (x, y), where x is an element of a and y is an element of b.

$$\frac{x_1 \in a_1 \wedge \ldots \wedge x_n \in a_n}{(x_1, \ldots, x_n) \in a_1 \times \ldots \times a_n} \text{ [tuple membership]}$$

Example

$$Days = \{ 1, 2, \ldots, 31 \}$$
$$Months = \{ January, February, \ldots, December \}$$
$$Years = \{ 1900, 1901, \ldots, 1999 \}$$

$$(22, November, 1990) \in Days \times Months \times Years$$

Ordered tuples

Two tuples are equal if they have exactly the same elements, and these are arranged in the same order:

$$\frac{x_1 = y_1 \wedge \ldots \wedge x_n = y_n}{(x_1, \ldots, x_n) = (y_1, \ldots, y_n)} \text{ [tuple equality]}$$

Component selection

We may select a component of an ordered tuple using a simple dot notation: if m is an integer between 1 and n, then

$$(x_1, x_2, .., x_n) \,.\, m = x_m$$

Example

$$date = (22, November, 1990)$$

$$date.1 = 22$$
$$date.2 = November$$
$$date.3 = 1990$$

Questions

$$\{ x : Person;\ y : Person \mid x \text{ is associated with } y \wedge x \text{ eats cheese} \wedge y \text{ reads Hello} \bullet x \}$$

- what if we are looking for both of them?
- what if we omit the term part?

Characteristic tuple

The expression

$$\{\, a : \mathbb{N};\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \land a \notin t \,\}$$

has the same value as

$$\{\, a : \mathbb{N};\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \land a \notin t \bullet (a, s, t) \,\}$$

Question

Can you exhibit an element of each of the following sets?

- $\{\, a : \mathbb{N};\ b : \mathbb{N} \mid a > b \,\}$
- $\{\, s : \mathbb{P}\,\mathbb{N};\ a : \mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \land a \in t \,\}$

Union

$$\frac{x \in (a \cup b)}{x \in a \lor x \in b}\ [\text{union}]$$

Example

$$\begin{array}{lcl} \mathit{Guests} & = & \text{Jim's friends} \\ & & \cup \\ & & \text{Eleanor's friends} \end{array}$$

Intersection

$$\frac{x \in (a \cap b)}{x \in a \land x \in b}\ [\text{intersection}]$$

Example

$$\mathit{Steve} \in \mathit{Choir} \cap \mathit{1st\ VIII} \cap \mathit{1st\ XV}$$

Difference

$$\frac{x \in (a \setminus b)}{x \in a \land x \notin b} \text{ [diff]}$$

Example

$$Jim \in Voters \setminus Conservatives$$

Distributed forms

$$\frac{x \in \bigcup s}{\exists\, a : s \bullet x \in a} \text{ [distributed union]}$$

$$\frac{x \in \bigcap s}{\forall\, a : s \bullet x \in a} \text{ [distributed intersection]}$$

Example

$$\begin{aligned} Europeans = \bigcup \{ & Belgians, British, Danes, Dutch, \\ & French, Germans, Greeks, Irish, \\ & Italians, Luxemburgers, Portuguese, \\ & Spanish, Swedes, \ldots \} \end{aligned}$$

$$\{Steve\} = \bigcap \{Choir, \mathit{1st\ VIII}, \mathit{1st\ XV}\}$$

Types

A type is a maximal set, at least within the confines of the current specification.

This ensures that each object x is associated with exactly one type: the largest set s present for which $x \in s$.

Integers

There is a single built in type, $\mathbb{Z}$, the type of all integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

Other types may be constructed from $\mathbb{Z}$, or from user-defined basic types.

Examples

- $\mathbb{Z} \times \mathbb{Z}$
- $\mathbb{P}\,\mathbb{Z}$
- *People*
- $\ldots$

Advantages

- adds structure and readability
- correctness via type-checking
- data types suggest implementation

Question

What is the type of the empty set?

Summary

- collections of objects
- membership
- extensionality
- comprehension
- $\mathbb{P}$ and $\times$
- types