Equality

Equality

- we identify expressions using the symbol $=$
- equalities form the atomic propositions in our logical language
- the only other way of obtaining an atomic proposition is through set membership

Examples

- $1 + 1 \; = \; 2$
- *Christmas Day* $=$ 25*th December*
- *Sellafield* $=$ *Windscale*

Axiom of reflection

$$\frac{}{t = t}\,[\text{eq-ref}]$$

- $1 + 1 \; = \; 1 + 1$

Symmetry

$$\frac{s = t}{t = s}$$

$$\frac{\text{the man who stole my idea} \;=\; \text{the man on the right}}{\text{the man on the right} \;=\; \text{the man who stole my idea}}$$

Transitivity

$$\frac{s = t \quad t = u}{s = u}$$

$$\frac{\ldots \quad \text{the man on the right} \;=\; \text{Professor Plum}}{\text{the man who stole my idea} \;=\; \text{Professor Plum}}$$

Substitution of equals

$$\frac{s = t \quad p[t / x]}{p[s / x]} \ [\text{eq-sub}]$$

$$\frac{25\textit{th December} \text{ falls on a Sunday this year}}{\textit{Christmas Day} \text{ falls on a Sunday this year}}$$

The one-point rule

If the identity of a bound variable is revealed within the quantified expression, we may replace all instances of that variable, and remove the existential quantifier.

$$\frac{\exists x : a \bullet p \land x = t}{t \in a \land p[t / x]} \ [\text{one-point rule}]$$

provided that x is not free in t

$$\frac{}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)}$$

$[t \in a \land p[t / x]]^{[1]}$

$$\frac{\dfrac{}{\exists x : a \bullet p \land x = t}}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)} \ [\Rightarrow\text{−intro}^{[1]}]$$

$[t \in a \land p[t / x]]^{[1]}$

$$\frac{\dfrac{\dfrac{}{t \in a} \qquad \dfrac{}{(p \land x = t)[t / x]}}{\exists x : a \bullet p \land x = t} \ [\exists\text{−intro}]}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)} \ [\Rightarrow\text{−intro}^{[1]}]$$

$[t \in a \land p[t / x]]^{[1]}$

$$\frac{\dfrac{\dfrac{[t \in a \land p[t / x]]^{[1]}}{t \in a} \ [\land\text{−elim1}] \qquad \dfrac{}{(p \land x = t)[t / x]}}{\exists x : a \bullet p \land x = t} \ [\exists\text{−intro}]}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)} \ [\Rightarrow\text{−intro}^{[1]}]$$

$[t \in a \wedge p[t / x]]^{[1]}$

$$\dfrac{\dfrac{\dfrac{[t \in a \wedge p[t / x]]^{[1]}}{t \in a}\,[\wedge\text{-elim1}] \qquad \dfrac{\dfrac{}{p[t / x] \wedge t = t}}{(p \wedge x = t)[t / x]}\,[\text{eq-sub}]}{\exists\, x : a \bullet p \wedge x = t}\,[\exists\text{-intro}]}{t \in a \wedge p[t / x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\,[\Rightarrow\text{-intro}^{[1]}]$$

$[t \in a \wedge p[t / x]]^{[1]}$

$$\dfrac{\dfrac{\dfrac{[t \in a \wedge p[t / x]]^{[1]}}{t \in a}\,[\wedge\text{-elim1}] \qquad \dfrac{\dfrac{\dfrac{}{p[t / x]} \quad \dfrac{}{t = t}}{p[t / x] \wedge t = t}\,[\wedge\text{-intro}]}{(p \wedge x = t)[t / x]}\,[\text{eq-sub}]}{\exists\, x : a \bullet p \wedge x = t}\,[\exists\text{-intro}]}{t \in a \wedge p[t / x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\,[\Rightarrow\text{-intro}^{[1]}]$$

$[t \in a \wedge p[t / x]]^{[1]}$

$$\dfrac{\dfrac{\dfrac{[t \in a \wedge p[t / x]]^{[1]}}{t \in a}\,[\wedge\text{-elim1}] \qquad \dfrac{\dfrac{\dfrac{[t \in a \wedge p[t / x]]^{[1]}}{p[t / x]}\,[\wedge\text{-elim2}] \quad \dfrac{}{t = t}}{p[t / x] \wedge t = t}\,[\wedge\text{-intro}]}{(p \wedge x = t)[t / x]}\,[\text{eq-sub}]}{\exists\, x : a \bullet p \wedge x = t}\,[\exists\text{-intro}]}{t \in a \wedge p[t / x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\,[\Rightarrow\text{-intro}^{[1]}]$$

$[t \in a \wedge p[t / x]]^{[1]}$

$$\dfrac{\dfrac{\dfrac{[t \in a \wedge p[t / x]]^{[1]}}{t \in a}\,[\wedge\text{-elim1}] \qquad \dfrac{\dfrac{\dfrac{[t \in a \wedge p[t / x]]^{[1]}}{p[t / x]}\,[\wedge\text{-elim2}] \quad \dfrac{}{t = t}\,[\text{eq-ref}]}{p[t / x] \wedge t = t}\,[\wedge\text{-intro}]}{(p \wedge x = t)[t / x]}\,[\text{eq-sub}]}{\exists\, x : a \bullet p \wedge x = t}\,[\exists\text{-intro}]}{t \in a \wedge p[t / x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\,[\Rightarrow\text{-intro}^{[1]}]$$

Question

What happens when we apply the one-point rule here?

- $\exists\, n : \mathbb{N} \bullet 4 + n = 6 \wedge n = 2$
- $\exists\, n : \mathbb{N} \bullet 6 + n = 4 \wedge n = -2$
- $\exists\, n : \mathbb{N} \bullet (\forall\, m : \mathbb{N} \bullet n > m) \wedge n = n + 1$

Uniqueness and quantity

- equality makes our predicate calculus more expressive
- we can formalise statements containing the phrases ‘at most’, and ‘no more than’
- there is a special notation for ‘there is a unique x such that’. We write

  $\exists_1\, x : A \bullet \ldots$

At most

$$\forall\, p, q, r : \mathit{Visitors} \bullet p = q \lor q = r \lor r = p$$

At least

$$\exists\, p, q : \mathit{Applicants} \bullet p \neq q$$

Exactly one

$$\exists\, b : \mathit{Book} \bullet b \in \mathit{Desk} \land (\forall\, c : \mathit{Book} \mid c \in \mathit{Desk} \bullet c = b)$$

$$\exists_1\, b : \mathit{Book} \bullet b \in \mathit{Desk}$$

Definite description

We may describe an object in terms of its properties without giving it a name.

We write $(\mu\, x : a \mid p)$ to denote the unique object x from a with property p.

μ expressions

$$\frac{\exists_1\, x : a \bullet p \quad t \in a \land p[t / x]}{t = (\mu\, x : a \mid p)}\ [\mu \text{ introduction}]$$

provided that x does not appear free in t

$$\frac{\exists_1\, x : a \bullet p \quad t = (\mu\, x : a \mid p)}{t \in a \land p[t / x]}\ [\mu \text{ elimination}]$$

provided that x does not appear free in t

Examples

- $(\mu\, x : \mathit{Person} \mid x$ shot John Lennon $)$
- $(\mu\, y : \mathit{Person} \mid y$ discovered radium $)$
- $(\mu\, z : \mathit{Colleges} \mid z$ is the oldest college in Oxford $)$

Question

Which of the following can you prove?

- $2 = (\mu\, n : \mathbb{N} \mid 4 + n = 6)$
- $3 \neq (\mu\, n : \mathbb{N} \mid 4 + n = 6)$
- $1 = (\mu\, n : \mathbb{N} \mid n = n + 0)$
- $1 = (\mu\, n : \mathbb{N} \mid n = n + 1)$

Generalised form

We write

$$(\mu\, x : a \mid p \bullet e)$$

to denote the expression e such that there is a unique x from a satisfying p.

Example

$$(\mu\, k : \mathit{Colleges} \mid k \text{ is the newest college in Oxford} \bullet \mathit{date\_of\_foundation}(k))$$

Summary

- equality
- reflexivity, symmetry, transitivity
- substitution of equals
- one-point rule
- μ expressions