Equality

Equality

- we identify expressions using the symbol $=$
- equalities form the atomic propositions in our logical language
- the only other way of obtaining an atomic proposition is through set membership

Examples

- $1 + 1 = 2$
- $\mathit{Christmas\ Day} = 25\mathit{th\ December}$
- $\mathit{Sellafield} = \mathit{Windscale}$

Axiom of reflection

$$\frac{}{t = t}\ [\text{eq-ref}]$$

- $1 + 1 = 1 + 1$

Symmetry

$$\frac{s = t}{t = s}$$

$$\frac{\text{the man who stole my idea} \;=\; \text{the man on the right}}{\text{the man on the right} \;=\; \text{the man who stole my idea}}$$

Transitivity

$$\frac{s = t \quad t = u}{s = u}$$

$$\frac{\ldots \quad \text{the man on the right} \;=\; \text{Professor Plum}}{\text{the man who stole my idea} \;=\; \text{Professor Plum}}$$

Substitution of equals

$$\frac{s = t \quad p[t / x]}{p[s / x]} \text{ [eq-sub]}$$

$$\frac{25th\,December \text{ falls on a Sunday this year}}{Christmas\,Day \text{ falls on a Sunday this year}}$$

The one-point rule

If the identity of a bound variable is revealed within the quantified expression, we may replace all instances of that variable, and remove the existential quantifier.

$$\frac{\exists\, x : a \bullet p \land x = t}{t \in a \land p[t / x]} \text{ [one-point rule]}$$

provided that x is not free in t

$$\frac{}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)}$$

$\lceil t \in a \land p[t / x] \rceil^{[1]}$

$$\dfrac{\dfrac{}{\exists x : a \bullet p \land x = t}}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)} \; [\Rightarrow\text{−intro}^{[1]}]$$

$$\lceil t \in a \wedge p[t / x] \rceil^{[1]}$$

$$\dfrac{\dfrac{\dfrac{}{t \in a} \qquad \dfrac{}{(p \wedge x = t)[t / x]}}{\exists\, x : a \bullet p \wedge x = t}\;[\exists\text{–intro}]}{t \in a \wedge p[t / x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\;[\Rightarrow\text{–intro}^{[1]}]$$

$$\lceil t \in a \wedge p[t / x] \rceil^{[1]}$$

$$\dfrac{\dfrac{\dfrac{\lceil t \in a \wedge p[t / x] \rceil^{[1]}}{t \in a}\;[\wedge\text{–elim1}] \qquad \dfrac{}{(p \wedge x = t)[t / x]}}{\exists\, x : a \bullet p \wedge x = t}\;[\exists\text{–intro}]}{t \in a \wedge p[t / x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\;[\Rightarrow\text{–intro}^{[1]}]$$

$[t \in a \land p[t / x]]^{[1]}$

$$\frac{\overline{\quad p[t / x] \land t = t \quad}}{(p \land x = t)[t / x]}\ [\text{eq-sub}]$$

$$\cfrac{\cfrac{\cfrac{[t \in a \land p[t / x]]^{[1]}}{t \in a}\ [\land\text{−elim1}]}{\exists x : a \bullet p \land x = t}\ [\exists\text{−intro}]}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)}\ [\Rightarrow\text{−intro}^{[1]}]$$

$[t \in a \land p[t / x]]^{[1]}$

$$\cfrac{\cfrac{\overline{p[t / x]} \qquad \overline{t = t}}{p[t / x] \land t = t}\ [\land\text{−intro}]}{(p \land x = t)[t / x]}\ [\text{eq-sub}]$$

$$\cfrac{\cfrac{\cfrac{[t \in a \land p[t / x]]^{[1]}}{t \in a}\ [\land\text{−elim1}]}{\exists x : a \bullet p \land x = t}\ [\exists\text{−intro}]}{t \in a \land p[t / x] \Rightarrow (\exists x : a \bullet p \land x = t)}\ [\Rightarrow\text{−intro}^{[1]}]$$

$$[t \in a \wedge p[t/x]]^{[1]}$$

$$\dfrac{\dfrac{[t \in a \wedge p[t/x]]^{[1]}}{p[t/x]}\;[\wedge\text{−elim2}] \qquad \dfrac{}{t = t}}{\dfrac{p[t/x] \wedge t = t}{(p \wedge x = t)[t/x]}\;[\text{eq-sub}]}\;[\wedge\text{−intro}]$$

$$\dfrac{\dfrac{\dfrac{[t \in a \wedge p[t/x]]^{[1]}}{t \in a}\;[\wedge\text{−elim1}]}{\exists\, x : a \bullet p \wedge x = t}\;[\exists\text{−intro}]}{t \in a \wedge p[t/x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\;[\Rightarrow\text{−intro}^{[1]}]$$

$$[t \in a \wedge p[t/x]]^{[1]}$$

$$\dfrac{\dfrac{[t \in a \wedge p[t/x]]^{[1]}}{p[t/x]}\;[\wedge\text{−elim2}] \qquad \dfrac{}{t = t}\;[\text{eq-ref}]}{\dfrac{p[t/x] \wedge t = t}{(p \wedge x = t)[t/x]}\;[\text{eq-sub}]}\;[\wedge\text{−intro}]$$

$$\dfrac{\dfrac{\dfrac{[t \in a \wedge p[t/x]]^{[1]}}{t \in a}\;[\wedge\text{−elim1}]}{\exists\, x : a \bullet p \wedge x = t}\;[\exists\text{−intro}]}{t \in a \wedge p[t/x] \Rightarrow (\exists\, x : a \bullet p \wedge x = t)}\;[\Rightarrow\text{−intro}^{[1]}]$$

Question

What happens when we apply the one-point rule here?

- $\exists\, n : \mathbb{N} \bullet 4 + n = 6 \wedge n = 2$
- $\exists\, n : \mathbb{N} \bullet 6 + n = 4 \wedge n = -2$
- $\exists\, n : \mathbb{N} \bullet (\forall\, m : \mathbb{N} \bullet n > m) \wedge n = n + 1$

Uniqueness and quantity

- equality makes our predicate calculus more expressive
- we can formalise statements containing the phrases ‘at most’, and ‘no more than’
- there is a special notation for ‘there is a unique x such that’. We write

$$\exists_1\, x : A \bullet \ldots$$

At most

$$\forall\, p, q, r : \mathit{Visitors} \bullet p = q \lor q = r \lor r = p$$

At least

$$\exists\, p, q : \mathit{Applicants} \bullet p \neq q$$

Exactly one

$$\exists\, b : Book \bullet b \in Desk \land (\forall\, c : Book \mid c \in Desk \bullet c = b)$$

$$\exists_1\, b : Book \bullet b \in Desk$$

Definite description

We may describe an object in terms of its properties without giving it a name.

We write $(\mu\, x : a \mid p)$ to denote the unique object x from a with property p.

μ expressions

$$\frac{\exists_1\, x : a \bullet p \qquad t \in a \land p[t / x]}{t = (\mu\, x : a \mid p)}\ [\mu \text{ introduction}]$$

provided that x does not appear free in t

$$\frac{\exists_1\, x : a \bullet p \qquad t = (\mu\, x : a \mid p)}{t \in a \land p[t / x]}\ [\mu \text{ elimination}]$$

provided that x does not appear free in t

Examples

- $(\ \mu\, x : \mathit{Person} \mid x$ shot John Lennon $)$
- $(\ \mu\, y : \mathit{Person} \mid y$ discovered radium $)$
- $(\ \mu\, z : \mathit{Colleges} \mid z$ is the oldest college in Oxford $)$

Question

Which of the following can you prove?

- $2 = (\mu\, n : \mathbb{N} \mid 4 + n = 6)$
- $3 \neq (\mu\, n : \mathbb{N} \mid 4 + n = 6)$
- $1 = (\mu\, n : \mathbb{N} \mid n = n + 0)$
- $1 = (\mu\, n : \mathbb{N} \mid n = n + 1)$

Generalised form

We write

$$(\mu\, x : a \mid p \bullet e)$$

to denote the expression e such that there is a unique x from a satisfying p.

Example

$$(\, \mu\, k : Colleges \mid k \text{ is the newest college in Oxford} \bullet \\ date\_of\_foundation(k)\,)$$

Summary

- equality
- reflexivity, symmetry, transitivity
- substitution of equals
- one-point rule
- μ expressions