Predicate logic

Predicates

A predicate is that part of a sentence which states something about the object of the sentence.

A predicate is a statement with a place for an object. When this place is filled, the predicate becomes a statement about the object that fills it.

A predicate is a proposition with a hole in it.

Variables

Instead of leaving a gap, as in

$$\_ > 5$$

we insert a variable

$$x > 5$$

Declarations

A statement such as $x > 5$ is not a proposition: its truth depends upon the value of variable x.

Before we can reason about such statements, we will need to declare, or introduce, the variables concerned.

The declaration $x : a$ introduces a variable x and tells us that it is an element of the set a.

Quantification

If p is a statement about x, then we may make it into a universal or existential statement by preceding it with a quantifier and a declaration.

$$\mathcal{Q}\, x : a \bullet p$$

Universal quantifier

Universal quantification:

$$\forall\, x : a \bullet p$$

'for all x in a, p holds'

Examples

Everybody has to do the assignment:

$$\forall\, s : \mathit{Student} \bullet s \text{ has to do the assignment}$$

Jim doesn't know anyone who can bail him out:

$$\forall\, p : \mathit{Person} \bullet \text{Jim knows } p \Rightarrow \neg\, p \text{ can bail Jim out}$$

Existential quantifier

Existential quantification:

$$\exists\, x : a \bullet p$$

'there exists an x in a such that p holds'

Examples

I heard it from one of your friends:

$\exists f : \mathit{Friends} \bullet \text{I heard it from } f$

A mad dog has bitten Andy:

$\exists d : \mathit{Dog} \bullet d \text{ is mad} \land d \text{ has bitten Andy}$

Constraints

We may add a predicate to the declaration part of a quantified expression to restrict the range of the variable.

$$\mathcal{Q}\, x : a \mid r \bullet p$$

In this expression, x ranges over those elements of a for which r is true.

Example

A constraint after ‘for all’ is an ‘only if’ clause:

$$(\forall\, x : a \mid r \bullet p) \;\Leftrightarrow\; (\forall\, x : a \bullet r \Rightarrow p)$$

Example

A constraint after 'there exists' is an additional conjunct:

$$(\exists\, x : a \mid r \bullet p) \;\Leftrightarrow\; (\exists\, x : a \bullet r \wedge p)$$

Free variables

In the expression $\mathcal{Q}\ x : a \mid r \bullet p$, we say that variable x is bound by the quantifier.

The scope of x extends from the vertical bar—or the spot, if there is no constraint—to the next enclosing bracket.

If variable x appears in a predicate q but is not bound by any quantifier, we say that x is free in q.

Example

There are free, bound, and binding occurrences of x in the predicate below:

$$x = 3 \;\wedge\; \forall\, x : \mathbb{N} \bullet 0 \leq x$$

Substitution

We write

$$p[t \,/\, x]$$

to denote the predicate that results from substituting t for each free occurrence of x in predicate p; this new operator binds more closely than any other.

Question

What happens here?

- $(x \leq y + 2)[0 \,/\, x] \;\Leftrightarrow\; \ldots$
- $(\exists\, x : \mathbb{N} \bullet x \leq y + 2)[0 \,/\, x] \;\Leftrightarrow\; \ldots$
- $(\exists\, x : \mathbb{N} \bullet x \leq y + 2)[5 \,/\, y] \;\Leftrightarrow\; \ldots$

Renaming bound variables

We may change the name of a bound variable without changing the meaning of an quantified statement, provided that the statement says nothing about the new name:

$$(\forall\, x : a \bullet p) \;\Leftrightarrow\; (\forall\, y : a \bullet p[y/x])$$

provided that y is not free in p

Example

There is no-one else who looks like *alan*:

$$\forall\, p : \mathit{Person} \bullet p \text{ looks like } \mathit{alan} \Rightarrow p = \mathit{alan}$$

Rename bound variable p to q:

$$\forall\, q : \mathit{Person} \bullet q \text{ looks like } \mathit{alan} \Rightarrow q = \mathit{alan}$$

Variable capture

The substitute expression t need not be another variable; it can be any expression whose possible values match those of x.

It may be necessary to rename bound and binding occurrences of other variables to avoid variable capture.

Example

There is no-one else who looks like *alan*:

$$\forall\, p : \mathit{Person} \bullet p \text{ looks like } \mathit{alan} \Rightarrow p = \mathit{alan}$$

Substitute *mike* for *alan*:

$$\forall\, p : \mathit{Person} \bullet p \text{ looks like } \mathit{mike} \Rightarrow p = \mathit{mike}$$

Substitute p for *alan*:

$$\forall\, p : \mathit{Person} \bullet p \text{ looks like } p \Rightarrow p = p$$

Conjunction

The universal quantifier is a generalised form of $\wedge$:

$$(0 > 5) \wedge (1 > 5) \wedge (2 > 5) \wedge (3 > 5) \wedge \ldots$$

$$\Leftrightarrow$$

$$\forall\, x : \mathbb{N} \bullet x > 5$$

Generalisation

$$\dfrac{\begin{array}{c}[x \in a]^{[i]} \\ \vdots \\ p\end{array}}{\forall\, x : a \bullet p}\ [\forall\!-\!\mathrm{intro}^{[i]}]$$

provided that x is not free in the assumptions of p

Specialisation

$$\frac{t \in a \qquad \forall\, x : a \bullet p}{p[t \,/\, x]}\ [\forall\!-\!\text{elim}]$$

Example

The statement

$$(\forall\, x : a \bullet p \wedge q) \Rightarrow (\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)$$

is a theorem of our natural deduction system.

$$(\forall\, x : a \bullet p \wedge q) \Rightarrow (\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)$$

$[\forall x : a \bullet p \wedge q]^{[1]}$

$$\dfrac{\dfrac{}{(\forall x : a \bullet p) \wedge (\forall x : a \bullet q)}}{(\forall x : a \bullet p \wedge q) \Rightarrow (\forall x : a \bullet p) \wedge (\forall x : a \bullet q)} \; [\Rightarrow\!-\mathrm{intro}^{[1]}]$$

$[\forall\, x : a \bullet p \wedge q]^{[1]}$

$$\dfrac{}{\forall\, x : a \bullet q}$$

$$\dfrac{\dfrac{}{\forall\, x : a \bullet p} \qquad \forall\, x : a \bullet q}{\dfrac{(\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)}{(\forall\, x : a \bullet p \wedge q) \Rightarrow (\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)}\ [\Rightarrow\text{−intro}^{[1]}]}\ [\wedge\text{−intro}]$$

$\lceil \forall x : a \bullet p \land q \rceil^{[1]} \qquad \lceil x \in a \rceil^{[2]}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p}}{\forall x : a \bullet p}\,[\forall{-}\mathrm{intro}^{[2]}] \qquad \dfrac{}{\forall x : a \bullet q}}{(\forall x : a \bullet p) \land (\forall x : a \bullet q)}\,[\land{-}\mathrm{intro}]}{(\forall x : a \bullet p \land q) \Rightarrow (\forall x : a \bullet p) \land (\forall x : a \bullet q)}\,[\Rightarrow{-}\mathrm{intro}^{[1]}]$$

$$\lceil \forall x : a \bullet p \land q \rceil^{[1]} \qquad \lceil x \in a \rceil^{[2]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \land q}}{p}\,[\land\text{−elim1}]}{\forall x : a \bullet p}\,[\forall\text{−intro}^{[2]}] \qquad \dfrac{}{\forall x : a \bullet q}}{(\forall x : a \bullet p) \land (\forall x : a \bullet q)}\,[\land\text{−intro}]}{(\forall x : a \bullet p \land q) \Rightarrow (\forall x : a \bullet p) \land (\forall x : a \bullet q)}\,[\Rightarrow\text{−intro}^{[1]}]$$

$[\forall\, x : a \bullet p \wedge q]^{[1]} \qquad [x \in a]^{[2]}$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x \in a]^{[2]} \quad [\forall\, x : a \bullet p \wedge q]^{[1]}}{p \wedge q}\,[\forall\text{-elim}]}{p}\,[\wedge\text{-elim1}]}{\forall\, x : a \bullet p}\,[\forall\text{-intro}^{[2]}] \qquad \dfrac{}{\forall\, x : a \bullet q}}{\dfrac{(\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)}{(\forall\, x : a \bullet p \wedge q) \Rightarrow (\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)}\,[\Rightarrow\text{-intro}^{[1]}]}\,[\wedge\text{-intro}]$$

$\lceil \forall\, x : a \bullet p \wedge q \rceil^{[1]} \qquad \lceil x \in a \rceil^{[2]} \qquad \lceil x \in a \rceil^{[3]}$

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        \lceil x \in a \rceil^{[2]} \quad \lceil \forall\, x : a \bullet p \wedge q \rceil^{[1]}
      }{p \wedge q}\;[\forall-\mathrm{elim}]
    }{p}\;[\wedge-\mathrm{elim1}]
  }{\forall\, x : a \bullet p}\;[\forall-\mathrm{intro}^{[2]}]
  \qquad
  \dfrac{
    \dfrac{}{q}
  }{\forall\, x : a \bullet q}\;[\forall-\mathrm{intro}^{[3]}]
}{
  \dfrac{(\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)}{(\forall\, x : a \bullet p \wedge q) \Rightarrow (\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)}\;[\Rightarrow-\mathrm{intro}^{[1]}]
}\;[\wedge-\mathrm{intro}]
$$

$$[\forall\, x : a \bullet p \land q]^{[1]} \qquad [x \in a]^{[2]} \qquad [x \in a]^{[3]}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[x \in a]^{[2]} \quad [\forall\, x : a \bullet p \land q]^{[1]}}{p \land q}\,[\forall-\mathrm{elim}]}{p}\,[\land-\mathrm{elim1}]}{\forall\, x : a \bullet p}\,[\forall-\mathrm{intro}^{[2]}] \qquad \cfrac{\cfrac{\cfrac{}{p \land q}}{q}\,[\land-\mathrm{elim2}]}{\forall\, x : a \bullet q}\,[\forall-\mathrm{intro}^{[3]}]}{(\forall\, x : a \bullet p) \land (\forall\, x : a \bullet q)}\,[\land-\mathrm{intro}]}{(\forall\, x : a \bullet p \land q) \Rightarrow (\forall\, x : a \bullet p) \land (\forall\, x : a \bullet q)}\,[\Rightarrow-\mathrm{intro}^{[1]}]$$

$$[\forall x : a \bullet p \land q]^{[1]} \qquad [x \in a]^{[2]} \qquad [x \in a]^{[3]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x \in a]^{[2]} \quad [\forall x : a \bullet p \land q]^{[1]}}{p \land q}\,[\forall{-}\mathrm{elim}]}{p}\,[\land{-}\mathrm{elim1}]}{\forall x : a \bullet p}\,[\forall{-}\mathrm{intro}^{[2]}] \qquad \dfrac{\dfrac{\dfrac{[x \in a]^{[3]} \quad [\forall x : a \bullet p \land q]^{[1]}}{p \land q}\,[\forall{-}\mathrm{elim}]}{q}\,[\land{-}\mathrm{elim2}]}{\forall x : a \bullet q}\,[\forall{-}\mathrm{intro}^{[3]}]}{(\forall x : a \bullet p) \land (\forall x : a \bullet q)}\,[\land{-}\mathrm{intro}]}{(\forall x : a \bullet p \land q) \Rightarrow (\forall x : a \bullet p) \land (\forall x : a \bullet q)}\,[\Rightarrow{-}\mathrm{intro}^{[1]}]$$

Disjunction

The existential quantifier is a generalised form of $\vee$:

$$(0 > 5) \vee (1 > 5) \vee (2 > 5) \vee (3 > 5) \vee \ldots$$

$$\Leftrightarrow$$

$$\exists\, x : \mathbb{N} \bullet x > 5$$

Introduction

$$\frac{t \in a \quad p[t\,/\,x]}{\exists\, x : a \bullet p}\ [\exists\text{−intro}]$$

Elimination

$$\dfrac{\exists\, x : a \bullet p \qquad \begin{array}{c} \lceil x \in a \rceil^{[i]} \\ \lceil p \rceil^{[i]} \\ \vdots \\ r \end{array}}{r} \; [\exists\text{−elim}^{[i]}]$$

provided that x is not free in the assumptions, and x is not free in r

Example

The statement

$$(\exists x : a \bullet \exists y : b \bullet p) \Rightarrow (\exists y : b \bullet \exists x : a \bullet p)$$

is a theorem of our natural deduction system, provided x is not free in b, and y is not free in a.

$$(\exists\, x : a \bullet \exists\, y : b \bullet p) \Rightarrow (\exists\, y : b \bullet \exists\, x : a \bullet p)$$

$$[\exists x : a \bullet \exists y : b \bullet p]^{[1]}$$

$$\dfrac{\dfrac{}{\exists y : b \bullet \exists x : a \bullet p}}{(\exists x : a \bullet \exists y : b \bullet p) \Rightarrow (\exists y : b \bullet \exists x : a \bullet p)}\ [\Rightarrow\!-\mathrm{intro}^{[1]}]$$

$$\lceil \exists x : a \bullet \exists y : b \bullet p \rceil^{[1]} \qquad \lceil x \in a \rceil^{[2]} \qquad \lceil \exists y : b \bullet p \rceil^{[2]}$$

$$\cfrac{\cfrac{\lceil \exists x : a \bullet \exists y : b \bullet p \rceil^{[1]} \qquad \cfrac{}{\exists y : b \bullet \exists x : a \bullet p}}{\exists y : b \bullet \exists x : a \bullet p}\ [\exists\text{–elim}^{[2]}]}{(\exists x : a \bullet \exists y : b \bullet p) \Rightarrow (\exists y : b \bullet \exists x : a \bullet p)}\ [\Rightarrow\text{–intro}^{[1]}]$$

$\lceil \exists x : a \bullet \exists y : b \bullet p \rceil^{[1]}$ $\lceil x \in a \rceil^{[2]}$ $\lceil \exists y : b \bullet p \rceil^{[2]}$

$\lceil y \in b \rceil^{[3]}$ $\lceil p \rceil^{[3]}$

$$\dfrac{\dfrac{\lceil \exists x : a \bullet \exists y : b \bullet p \rceil^{[1]} \qquad \dfrac{\lceil \exists y : b \bullet p \rceil^{[2]} \qquad \dfrac{}{\exists y : b \bullet \exists x : a \bullet p}}{\exists y : b \bullet \exists x : a \bullet p}\ [\exists\text{–elim}^{[3]}]}{\exists y : b \bullet \exists x : a \bullet p}\ [\exists\text{–elim}^{[2]}]}{(\exists x : a \bullet \exists y : b \bullet p) \Rightarrow (\exists y : b \bullet \exists x : a \bullet p)}\ [\Rightarrow\text{–intro}^{[1]}]$$

$\lceil \exists x : a \bullet \exists y : b \bullet p \rceil^{[1]} \qquad \lceil x \in a \rceil^{[2]} \qquad \lceil \exists y : b \bullet p \rceil^{[2]}$

$\lceil y \in b \rceil^{[3]} \qquad \lceil p \rceil^{[3]}$

$$\dfrac{\lceil y \in b \rceil^{[3]} \qquad \dfrac{}{\exists x : a \bullet p}}{\exists y : b \bullet \exists x : a \bullet p}\ [\exists\text{−intro}]$$

$$\dfrac{\lceil \exists y : b \bullet p \rceil^{[2]} \qquad \exists y : b \bullet \exists x : a \bullet p}{\exists y : b \bullet \exists x : a \bullet p}\ [\exists\text{−elim}^{[3]}]$$

$$\dfrac{\dfrac{\lceil \exists x : a \bullet \exists y : b \bullet p \rceil^{[1]} \qquad \exists y : b \bullet \exists x : a \bullet p}{\exists y : b \bullet \exists x : a \bullet p}\ [\exists\text{−elim}^{[2]}]}{(\exists x : a \bullet \exists y : b \bullet p) \Rightarrow (\exists y : b \bullet \exists x : a \bullet p)}\ [\Rightarrow\text{−intro}^{[1]}]$$

$$\lceil \exists\, x : a \bullet \exists\, y : b \bullet p \rceil^{[1]} \qquad \lceil x \in a \rceil^{[2]} \qquad \lceil \exists\, y : b \bullet p \rceil^{[2]}$$

$$\lceil y \in b \rceil^{[3]} \qquad \lceil p \rceil^{[3]}$$

$$\dfrac{\lceil y \in b \rceil^{[3]} \qquad \dfrac{\lceil x \in a \rceil^{[2]} \qquad \lceil p \rceil^{[3]}}{\exists\, x : a \bullet p}\;[\exists\text{–intro}]}{\exists\, y : b \bullet \exists\, x : a \bullet p}\;[\exists\text{–intro}]$$

$$\dfrac{\lceil \exists\, y : b \bullet p \rceil^{[2]} \qquad \exists\, y : b \bullet \exists\, x : a \bullet p}{\exists\, y : b \bullet \exists\, x : a \bullet p}\;[\exists\text{–elim}^{[3]}]$$

$$\dfrac{\dfrac{\lceil \exists\, x : a \bullet \exists\, y : b \bullet p \rceil^{[1]} \qquad \exists\, y : b \bullet \exists\, x : a \bullet p}{\exists\, y : b \bullet \exists\, x : a \bullet p}\;[\exists\text{–elim}^{[2]}]}{(\exists\, x : a \bullet \exists\, y : b \bullet p) \Rightarrow (\exists\, y : b \bullet \exists\, x : a \bullet p)}\;[\Rightarrow\text{–intro}^{[1]}]$$

Summary

- predicates
- quantifiers
- bound variables
- substitution
- $\forall$-introduction and elimination
- $\exists$-introduction and elimination