# Propositional logic



## Propositional logic

- deals with propositions: statements that must be either true or false
- propositions may be combined using logical connectives
- the meaning of a combination is determined by the meanings of the propositions involved

## Atomic propositions

Atomic propositions are statements without logical connectives.

In our language, an atomic proposition will state either

- that an object is a member of a set, or
- that it is equal to another object.

We will see how to formalise these statements later.



## Examples

- jaffa cakes are biscuits
- your cat is rich
- your dog is good looking
- $2 + 2 = 5$
- $tomorrow = tuesday$

## Connectives

| | |
|---|---|
| $\neg$ | negation (not) |
| $\wedge$ | conjunction (and) |
| $\vee$ | disjunction (or) |
| $\Rightarrow$ | implication (implies or only if) |
| $\Leftrightarrow$ | equivalence (iff or if and only if) |



## Examples

- $\neg$ (jaffa cakes are biscuits)
- your cat is rich $\wedge$ your dog is good looking
- the map is wrong $\vee$ you are a poor navigator
- $(2 + 2 = 5) \Rightarrow$ (unemployment $<$ 2 million)
- $(\textit{tomorrow} = \textit{tuesday}) \Leftrightarrow (\textit{today} = \textit{monday})$

## Truth tables

We use truth tables to give a precise meaning to our logical connectives.

The following table (completed) would give the meaning of the (imaginary) connective $\diamond$:

| $p$ | $q$ | $p \diamond q$ |
|---|---|---|
| $t$ | $t$ | |
| $t$ | $f$ | |
| $f$ | $t$ | |
| $f$ | $f$ | |



## Example

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| $t$ | $t$ | $t$ |
| $t$ | $f$ | $f$ |
| $f$ | $t$ | $f$ |
| $f$ | $f$ | $f$ |

## Question

How should we complete the following?

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| $t$ | $t$ | |
| $t$ | $f$ | |
| $f$ | $t$ | |
| $f$ | $f$ | |



## Question

How should we complete the following?

| $p$ | $q$ | $p \Rightarrow q$ |
|---|---|---|
| $t$ | $t$ | |
| $t$ | $f$ | |
| $f$ | $t$ | |
| $f$ | $f$ | |

## Question

How should we complete the following?

| $p$ | $q$ | $p \Leftrightarrow q$ |
|---|---|---|
| $t$ | $t$ | |
| $t$ | $f$ | |
| $f$ | $t$ | |
| $f$ | $f$ | |



## Question

How should we complete the following?

| $p$ | $\neg\, p$ |
|---|---|
| $t$ | |
| $f$ | |

## Precedence

Negation binds more closely than conjunction:

| | |
|---|---|
| $\neg$ | highest |
| $\wedge$ | |
| $\vee$ | |
| $\Rightarrow$ | |
| $\Leftrightarrow$ | lowest |



## Example

The proposition

$$\neg\, p \wedge q \vee r \Leftrightarrow q \Rightarrow p \wedge r$$

is equivalent to

$$(((\neg\, p) \wedge q) \vee r) \Leftrightarrow (q \Rightarrow (p \wedge r))$$

## Inference rules

To construct arguments about propositions, we use a system of natural deduction: a collection of inference rules.

The following rule states that whenever all of the premisses hold, then the conclusion must be true.

$$\frac{\textit{premisses}}{\textit{conclusion}}\ [\text{name}] \qquad \text{side condition}$$

It may be used only when the side condition is true.



## Names

Each of our basic rules, or axioms, is associated with a particular connective.

An introduction rule has a fresh instance of the connective within its conclusion.

An elimination rule has an instance of the connective within one of its premisses.

## Conjunction

and-introduction:

$$\frac{p \quad q}{p \wedge q}\ [\wedge\text{−intro}]$$

and-elimination:

$$\frac{p \wedge q}{p}\ [\wedge\text{−elim1}] \qquad \frac{p \wedge q}{q}\ [\wedge\text{−elim2}]$$



## Arguments

We use a similar syntax—proof trees—to present arguments or derivations.

A derivation shows how a conclusion may be reached from a set of premisses.

If the set of premisses is empty—that is, if none are required—then the conclusion is said to be a theorem.

If an argument is valid, then it is possible to justify every step in terms of the basic rules.

## Example

The following (trivial) argument states that whenever $p \wedge q$ is true, then so too is $q \wedge p$:

$$\frac{p \wedge q}{q \wedge p}$$

We can expand upon this:

$$\cfrac{\cfrac{p \wedge q}{q}\,[\wedge\text{-elim2}] \quad \cfrac{p \wedge q}{p}\,[\wedge\text{-elim1}]}{q \wedge p}\,[\wedge\text{-intro}]$$



## Assumptions

In the course of an argument, we may assume temporarily that a particular statement is true.

Such an assumption must be allowed (and discharged) by an appropriate inference rule.

An assumption can be used repeatedly throughout its scope.

## Presentation

In a proof tree, the scope of an assumption is a single sub-tree, or branch, extending upwards from the rule that allows it.

The step at which an assumption (or set of assumptions) is discharged will be numbered. This number will be used to identify the assumption (or set of assumptions) concerned.

We write $\lceil p \rceil^{[i]}$ to denote the assumption of statement $p$, labelled with number $i$.



## Disjunction

*or*-introduction:

$$\frac{p}{p \vee q}\,[\vee\text{–intro1}] \qquad\qquad \frac{q}{p \vee q}\,[\vee\text{–intro2}]$$

*or*-elimination:

$$\frac{p \vee q \qquad \begin{matrix}\lceil p \rceil^{[i]} \\ r\end{matrix} \qquad \begin{matrix}\lceil q \rceil^{[i]} \\ r\end{matrix}}{r}\,[\vee\text{–elim}^{[i]}]$$

## Example

If $p \vee q$ is true, then $q \vee p$ must be true also:

$$\dfrac{p \vee q \qquad \dfrac{[p]^{[1]}}{q \vee p}\,[\vee\text{–intro2}] \qquad \dfrac{[q]^{[1]}}{q \vee p}\,[\vee\text{–intro1}]}{q \vee p}\,[\vee\text{–elim}^{[1]}]$$



## Example

The following argument is not valid:

$$\dfrac{p \vee q \qquad \begin{array}{c}[p \wedge \neg q]^{[i]} \\ r\end{array} \qquad \begin{array}{c}[q \wedge \neg p]^{[i]} \\ r\end{array}}{r}\,[\vee\text{–elim}'^{[i]}]$$

"I've pressed A or B. Now, A gives me coffee and B gives me tea.
So I've got a hot drink."

## Implication

⇒-introduction:

$$\dfrac{\begin{array}{c}[p]^{[i]}\\ q\end{array}}{p \Rightarrow q}\ [\Rightarrow\text{–intro}^{[i]}]$$

⇒-elimination:

$$\dfrac{p \Rightarrow q \quad p}{q}\ [\Rightarrow\text{–elim}]$$



## Example

The statement

$$(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))$$

is a theorem of our natural deduction system.

$$\frac{}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}$$



$$\lceil p \wedge q \Rightarrow r \rceil^{[1]}$$

$$\frac{\dfrac{}{(p \Rightarrow (q \Rightarrow r))}}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\!-\text{intro}^{[1]}]$$

$$[p \land q \Rightarrow r]^{[1]} \qquad [p]^{[2]}$$

$$\dfrac{\dfrac{\dfrac{}{q \Rightarrow r}}{(p \Rightarrow (q \Rightarrow r))}\;[\Rightarrow\text{–intro}^{[2]}]}{(p \land q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\;[\Rightarrow\text{–intro}^{[1]}]$$



$$[p \land q \Rightarrow r]^{[1]} \qquad [p]^{[2]} \qquad [q]^{[3]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{r}}{q \Rightarrow r}\;[\Rightarrow\text{–intro}^{[3]}]}{(p \Rightarrow (q \Rightarrow r))}\;[\Rightarrow\text{–intro}^{[2]}]}{(p \land q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\;[\Rightarrow\text{–intro}^{[1]}]$$

$[p \wedge q \Rightarrow r]^{[1]} \qquad [p]^{[2]} \qquad [q]^{[3]}$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p \wedge q \Rightarrow r]^{[1]} \qquad \dfrac{}{p \wedge q}}{r}\,[\Rightarrow\text{-elim}]}{q \Rightarrow r}\,[\Rightarrow\text{-intro}^{[3]}]}{(p \Rightarrow (q \Rightarrow r))}\,[\Rightarrow\text{-intro}^{[2]}]}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\,[\Rightarrow\text{-intro}^{[1]}]$$



$[p \wedge q \Rightarrow r]^{[1]} \qquad [p]^{[2]} \qquad [q]^{[3]}$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p \wedge q \Rightarrow r]^{[1]} \qquad \dfrac{[p]^{[2]} \quad [q]^{[3]}}{p \wedge q}\,[\wedge\text{-intro}]}{r}\,[\Rightarrow\text{-elim}]}{q \Rightarrow r}\,[\Rightarrow\text{-intro}^{[3]}]}{(p \Rightarrow (q \Rightarrow r))}\,[\Rightarrow\text{-intro}^{[2]}]}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\,[\Rightarrow\text{-intro}^{[1]}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[p \wedge q \Rightarrow r]^{[1]} \qquad \dfrac{[p]^{[2]} \quad [q]^{[3]}}{p \wedge q}\ [\wedge\text{–intro}]}{r}\ [\Rightarrow\text{–elim}]}{q \Rightarrow r}\ [\Rightarrow\text{–intro}^{[3]}]}{(p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\text{–intro}^{[2]}]}{(p \wedge q \Rightarrow r) \Rightarrow (p \Rightarrow (q \Rightarrow r))}\ [\Rightarrow\text{–intro}^{[1]}]$$



## Example

The following argument is not valid:

$$\dfrac{p \Rightarrow q \quad q}{p}\ [\Rightarrow\text{–elim}]$$

"if it's Wednesday, then I'm in Guildford and I'm in Guildford, so it's Wednesday"

## Transitivity

Implication is transitive: that is,

$$\frac{p \Rightarrow q \quad q \Rightarrow r}{p \Rightarrow r}$$



## Presentation

We write

$$\cfrac{\cfrac{\cfrac{}{a \Rightarrow b}\,[\text{reason 1}] \quad \cfrac{}{b \Rightarrow c}\,[\text{reason 2}]}{a \Rightarrow c} \qquad \cfrac{}{c \Rightarrow d}\,[\text{reason 3}]}{a \Rightarrow d}$$

as

$$\begin{array}{ll} a & \\ \Rightarrow b & [\text{reason 1}] \\ \Rightarrow c & [\text{reason 2}] \\ \Rightarrow d & [\text{reason 3}] \end{array}$$

## Equivalence

$\Leftrightarrow$-introduction:

$$\frac{p \Rightarrow q \quad q \Rightarrow p}{p \Leftrightarrow q}\ [\Leftrightarrow\text{–intro}]$$

$\Leftrightarrow$-elimination:

$$\frac{p \Leftrightarrow q}{p \Rightarrow q}\ [\Leftrightarrow\text{–elim1}] \qquad\qquad \frac{p \Leftrightarrow q}{q \Rightarrow p}\ [\Leftrightarrow\text{–elim2}]$$



## Question

How could we show that

$$\frac{p \Rightarrow q}{p \wedge q \Leftrightarrow p}$$

is a valid inference in our natural deduction system?

$$\dfrac{}{p \wedge q \Leftrightarrow p}$$



$$\dfrac{\dfrac{}{p \wedge q \Rightarrow p} \qquad \dfrac{}{p \Rightarrow p \wedge q}}{p \wedge q \Leftrightarrow p}\;[\Leftrightarrow\text{–intro}]$$

$[p \wedge q]^{[1]}$

$$
\cfrac{\cfrac{\cfrac{}{p}}{p \wedge q \Rightarrow p}\,[\Rightarrow\text{–intro}^{[1]}] \qquad \cfrac{}{p \Rightarrow p \wedge q}}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{–intro}]
$$



$[p \wedge q]^{[1]}$

$$
\cfrac{\cfrac{\cfrac{[p \wedge q]^{[1]}}{p}\,[\wedge\text{–elim1}]}{p \wedge q \Rightarrow p}\,[\Rightarrow\text{–intro}^{[1]}] \qquad \cfrac{}{p \Rightarrow p \wedge q}}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{–intro}]
$$

$\lceil p \wedge q \rceil^{[1]} \qquad \lceil p \rceil^{[2]}$

$$\dfrac{\dfrac{\dfrac{\lceil p \wedge q \rceil^{[1]}}{p}\,[\wedge\text{−elim1}]}{p \wedge q \Rightarrow p}\,[\Rightarrow\text{−intro}^{[1]}] \qquad \dfrac{\dfrac{}{p \wedge q}}{p \Rightarrow p \wedge q}\,[\Rightarrow\text{−intro}^{[2]}]}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{−intro}]$$



$\lceil p \wedge q \rceil^{[1]} \qquad \lceil p \rceil^{[2]}$

$$\dfrac{\dfrac{\dfrac{\lceil p \wedge q \rceil^{[1]}}{p}\,[\wedge\text{−elim1}]}{p \wedge q \Rightarrow p}\,[\Rightarrow\text{−intro}^{[1]}] \qquad \dfrac{\dfrac{\lceil p \rceil^{[2]} \quad \dfrac{}{q}}{p \wedge q}\,[\wedge\text{−intro}]}{p \Rightarrow p \wedge q}\,[\Rightarrow\text{−intro}^{[2]}]}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{−intro}]$$

$[p \wedge q]^{[1]} \qquad [p]^{[2]}$

$$\dfrac{\dfrac{\dfrac{[p \wedge q]^{[1]}}{p}\,[\wedge\text{−elim1}]}{p \wedge q \Rightarrow p}\,[\Rightarrow\text{−intro}^{[1]}] \qquad \dfrac{\dfrac{[p]^{[2]} \quad \dfrac{p \Rightarrow q \quad [p]^{[2]}}{q}\,[\Rightarrow\text{−elim}]}{p \wedge q}\,[\wedge\text{−intro}]}{p \Rightarrow p \wedge q}\,[\Rightarrow\text{−intro}^{[2]}]}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{−intro}]$$



$$\dfrac{\dfrac{\dfrac{[p \wedge q]^{[1]}}{p}\,[\wedge\text{−elim1}]}{p \wedge q \Rightarrow p}\,[\Rightarrow\text{−intro}^{[1]}] \qquad \dfrac{\dfrac{[p]^{[2]} \quad \dfrac{p \Rightarrow q \quad [p]^{[2]}}{q}\,[\Rightarrow\text{−elim}]}{p \wedge q}\,[\wedge\text{−intro}]}{p \Rightarrow p \wedge q}\,[\Rightarrow\text{−intro}^{[2]}]}{p \wedge q \Leftrightarrow p}\,[\Leftrightarrow\text{−intro}]$$

## False

In reasoning about statements involving negation, it is often useful to consider the special proposition *false*, which is always false.



## Negation

*false*-elimination:

$$\cfrac{\begin{array}{c}[\neg\, p]^{[i]}\\ \mathit{false}\end{array}}{p}\ [\mathit{false}\text{–elim1}^{[i]}] \qquad \cfrac{\begin{array}{c}[p]^{[i]}\\ \mathit{false}\end{array}}{\neg\, p}\ [\mathit{false}\text{–elim2}^{[i]}]$$

*false*-introduction:

$$\frac{p \quad \neg\, p}{\mathit{false}}\ [\mathit{false}\text{–intro}]$$

## Example

The statement

$$p \vee \neg\, p$$

is a theorem of our natural deduction system.



$$\frac{}{p \vee \neg\, p}$$

$\lceil \neg\, (p \vee \neg\, p) \rceil^{[1]}$

$$\dfrac{\dfrac{\qquad\qquad\qquad}{\mathit{false}}}{p \vee \neg\, p}\ [\mathit{false}\text{–elim1}^{[1]}]$$



$\lceil \neg\, (p \vee \neg\, p) \rceil^{[1]}$

$$\dfrac{\dfrac{\dfrac{}{p} \qquad \dfrac{}{\neg\, p}}{\mathit{false}}\ [\mathit{false}\text{–intro}]}{p \vee \neg\, p}\ [\mathit{false}\text{–elim1}^{[1]}]$$

$\lceil \neg\, (p \vee \neg\, p) \rceil^{[1]} \qquad \lceil \neg\, p \rceil^{[2]}$

$$\cfrac{\cfrac{\cfrac{}{false}}{p}\;[false\text{–elim1}^{[2]}] \qquad \cfrac{}{\neg\, p}}{\cfrac{false}{p \vee \neg\, p}\;[false\text{–elim1}^{[1]}]}\;[false\text{–intro}]$$



$\lceil \neg\, (p \vee \neg\, p) \rceil^{[1]} \qquad \lceil \neg\, p \rceil^{[2]}$

$$\cfrac{\cfrac{\cfrac{\lceil \neg\, (p \vee \neg\, p) \rceil^{[1]} \quad \cfrac{}{p \vee \neg\, p}}{false}\;[false\text{–intro}]}{p}\;[false\text{–elim1}^{[2]}] \qquad \cfrac{}{\neg\, p}}{\cfrac{false}{p \vee \neg\, p}\;[false\text{–elim1}^{[1]}]}\;[false\text{–intro}]$$

$$\lceil \neg (p \vee \neg p) \rceil^{[1]} \qquad \lceil \neg p \rceil^{[2]}$$

$$\dfrac{}{\neg p}$$

$$\dfrac{\dfrac{\dfrac{\lceil \neg (p \vee \neg p) \rceil^{[1]} \quad \dfrac{\lceil \neg p \rceil^{[2]}}{p \vee \neg p}\,[\vee\text{–intro2}]}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p}\,[\mathit{false}\text{–elim1}^{[2]}]}{\dfrac{\mathit{false}}{p \vee \neg p}\,[\mathit{false}\text{–elim1}^{[1]}]}\,[\mathit{false}\text{–intro}]$$



$$\lceil \neg (p \vee \neg p) \rceil^{[1]} \qquad \lceil \neg p \rceil^{[2]} \qquad \lceil p \rceil^{[3]}$$

$$\dfrac{\dfrac{}{\mathit{false}}}{\neg p}\,[\mathit{false}\text{–elim1}^{[3]}]$$

$$\dfrac{\dfrac{\dfrac{\lceil \neg (p \vee \neg p) \rceil^{[1]} \quad \dfrac{\lceil \neg p \rceil^{[2]}}{p \vee \neg p}}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p}\,[\mathit{false}\text{–elim1}^{[2]}]}{\dfrac{\mathit{false}}{p \vee \neg p}\,[\mathit{false}\text{–elim1}^{[1]}]}\,[\mathit{false}\text{–intro}]$$

$[\neg (p \vee \neg p)]^{[1]} \qquad [\neg p]^{[2]} \qquad [p]^{[3]}$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg (p \vee \neg p)]^{[1]} \quad \dfrac{}{p \vee \neg p}}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{\neg p}\,[\mathit{false}\text{–elim1}^{[3]}] \\ \dfrac{\dfrac{[\neg (p \vee \neg p)]^{[1]} \quad \dfrac{[\neg p]^{[2]}}{p \vee \neg p}}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p}\,[\mathit{false}\text{–elim1}^{[2]}]\end{array}}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p \vee \neg p}\,[\mathit{false}\text{–elim1}^{[1]}]$$



$[\neg (p \vee \neg p)]^{[1]} \qquad [\neg p]^{[2]} \qquad [p]^{[3]}$

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[\neg (p \vee \neg p)]^{[1]} \quad \dfrac{[p]^{[3]}}{p \vee \neg p}\,[\vee\text{–intro1}]}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{\neg p}\,[\mathit{false}\text{–elim1}^{[3]}] \\ \dfrac{\dfrac{[\neg (p \vee \neg p)]^{[1]} \quad \dfrac{[\neg p]^{[2]}}{p \vee \neg p}\,[\vee\text{–intro2}]}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p}\,[\mathit{false}\text{–elim1}^{[2]}]\end{array}}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p \vee \neg p}\,[\mathit{false}\text{–elim1}^{[1]}]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg\,(p \vee \neg\,p)]^{[1]} \qquad \dfrac{[p]^{[3]}}{p \vee \neg\,p}\,[\vee\text{–intro1}]}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{\neg\,p}\,[\mathit{false}\text{–elim1}^{[3]}] \qquad \dfrac{\dfrac{[\neg\,(p \vee \neg\,p)]^{[1]} \qquad \dfrac{[\neg\,p]^{[2]}}{p \vee \neg\,p}\,[\vee\text{–intro2}]}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p}\,[\mathit{false}\text{–elim1}^{[2]}]}{\mathit{false}}\,[\mathit{false}\text{–intro}]}{p \vee \neg\,p}\,[\mathit{false}\text{–elim1}^{[1]}]$$



## Lemmas

We may use one argument as part of another. In this case, we might describe the subsidiary result as a lemma.

## Example

We might use the theorem

$$\frac{}{p \vee \neg p}$$

to help expand the following argument:

$$\frac{\neg (p \wedge q)}{\neg p \vee \neg q}$$



## Tautologies

A proposition that is true whether its components are true or false is said to be a tautology.

Our deduction system is complete, which means that every tautology is a theorem.

## Question

Which of the following are tautologies?

- $(p \wedge \neg p) \Rightarrow p$
- $(p \vee \neg p) \Rightarrow \neg p$
- $p \Rightarrow (q \Rightarrow p)$
- $(p \Rightarrow q) \Rightarrow p$



## Implications

If the implication

$$\mathit{proposition}1 \Rightarrow \mathit{proposition}2$$

is a tautology, then

$$\frac{\mathit{proposition}1}{\mathit{proposition}2}$$

is a valid argument.

## Example

$$p \Rightarrow (q \Rightarrow p)$$

$$\frac{p}{q \Rightarrow p}$$



## Equivalences

If the equivalence

$$\mathit{proposition}1 \Leftrightarrow \mathit{proposition}2$$

is a tautology, then we may replace instances of *proposition*1 in an expression with instances of *proposition*2.

## Examples

contrapositive:

$$p \Rightarrow q \quad \Leftrightarrow \quad \neg\, q \Rightarrow \neg\, p$$

de Morgan:

$$\neg\,(p \wedge q) \quad \Leftrightarrow \quad \neg\, p \vee \neg\, q$$



## Summary

- propositions
- $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$
- truth tables
- inference rules
- assumptions
- tautologies