Relations

Exercise 7.1 (Extension) Write the following relations as sets in extension—that is, as a list of elements between braces—

(a) $\{0, 1\} \leftrightarrow \{0, 1\}$

(b) $\{0\} \leftrightarrow \{0, 1\}$

(c) $\varnothing \leftrightarrow \{0, 1\}$

(d) $\varnothing \leftrightarrow (\varnothing \leftrightarrow \varnothing)$

□

Exercise 7.2 (Domain and range) R is the relation

$$\{0 \mapsto 1, 0 \mapsto 2, 0 \mapsto 3, 1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3\}$$

Write the following sets in extension:

(a) $\operatorname{dom} R$

(b) $\operatorname{ran} R$

(c) $\{1, 2\} \lhd R$

(d) $R \rhd \{0\}$

□

Exercise 7.3 (Domain and range restriction) If R, S, A, B, and C are defined by

$$\begin{array}{l} R, S : X \leftrightarrow Y \\ A, B : \mathbb{P}\, X \\ C : \mathbb{P}\, Y \end{array}$$

show that

(a) $A \lhd (B \lhd R) = (A \cap B) \lhd R$

(b) $(R \cup S) \rhd C = (R \rhd C) \cup (S \rhd C)$

(c) $(A \setminus B) \lhd R = (A \lhd R) \setminus (B \lhd R)$

□

Exercise 7.4 (Relatives) If a and b are elements of the set *Person*, then we write $a \mathit{\ child\ } b$ to indicate that a is a child of b. Using the relation *child*, define

(a) the relation *parent*, which holds between two people exactly when the first is the parent of the second.

(b) the relation *sibling*, which holds between two people exactly when they have the same parent.

(c) the relation *cousin*, which holds between two people exactly when a parent of one is a sibling of the parent of the other.

(d) the relation *ancestor*, which holds between two people exactly when the first is an ancestor of the second.

□

Exercise 7.5 (More relatives) Each person has exactly two genetic parents, one drawn from the set *Male* and the other drawn from the set *Female*. For the purposes of this example, we will assume that these are disjoint sets such that

$$\mathit{Person} = \mathit{Male} \cup \mathit{Female}$$

If *child* is as defined above, find suitable expressions for

(a) the relation *mother*, which holds between two people exactly when the first is the female parent of the second.

(b) the set *grandmothers*, which consists of all those people who are, or were, female grandparents.

(c) the relation *only_son*, which holds between two people exactly when the first is the only male child of the second.

(d) the relation *kissing*, which holds between two cousins exactly when one is male and the other is female.

□

Exercise 7.6 (Composition) The relations R and S are defined by

$$R == \{1 \mapsto 4, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 1\}$$
$$S == \{1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4\}$$

Write each of the following sets in extension:

(a) $R \fatsemi S$

(b) $S \fatsemi R$

(c) $R \fatsemi R$

(d) $R \fatsemi R \fatsemi R$

□

Exercise 7.7 (Find the relations) Find relations R, S, and T in the set $\{0, 1\} \leftrightarrow \{0, 1\}$ such that

(a) $R \fatsemi R = \mathrm{id}[\{0, 1\}]$ but $R \neq \mathrm{id}[\{0, 1\}]$

(b) $S \fatsemi S^{\sim} \fatsemi S = S$ but $S \neq S^{\sim}$

(c) $T \fatsemi T = \varnothing$ but $T \neq \varnothing$

□

Exercise 7.8 (Closures) If relations R and S are defined by

$$R == \{ n : \mathbb{N} \bullet n \mapsto n + 1 \}$$
$$S == \{ m : \mathbb{Z} \bullet m \mapsto |m| \}$$

identify the reflexive, symmetric, transitive, and reflexive transitive closures of each relation. □

Exercise 7.9 (Transitivity and subset) The property of *transitivity* could be defined by either of the following equations:

$$Transitive[X] = \{ R : X \leftrightarrow X \mid \forall x, y, z : X \bullet x \mapsto y \in R \land y \mapsto z \in R \Rightarrow x \mapsto z \in R \} \tag{7.1}$$

$$Transitive[X] = \{ R : X \leftrightarrow X \mid R \fatsemi R \subseteq R \} \tag{7.2}$$

Show that these two equations describe the same set: that is, demonstrate that the right-hand side of 7.1 is equal to the right-hand side of 7.2. We may then use either set in place of $Transitive[X]$. □