Exercises for Chapter 5

Sets

Exercise 5.1 (Membership) Which of the following statements are true?

(a) $1 \in \{4, 3, 2, 1\}$

(b) $\{1\} \in \{1, 2, 3, 4\}$

(c) $\{1\} \in \{\{1\}, \{2\}, \{3\}, \{4\}\}$

(d) $\emptyset \in \{1, 2, 3, 4\}$

□

Exercise 5.2 (Comprehension) Given that $*$ denotes multiplication, that `div` denotes integer division, and `mod` denotes remainder, describe each of the following using set comprehension:

(a) the set of integers between 0 and 100 inclusive

(b) the set of integer multiples of 10

(c) the set of integers divisible by 2, 3, and 5

□

Exercise 5.3 (Power sets) List *all* of the elements of

(a) $\mathbb{P}\{1, 2\}$

(b) $\mathbb{P}\,\mathbb{P}\{1, 2\}$

(c) $\mathbb{P}\,\varnothing$

(d) $\mathbb{P}\,\mathbb{P}\,\varnothing$

□

Exercise 5.4 (Cartesian products) List *all* of the elements of

(a) $\{1\} \times \{2, 3\}$

(b) $\varnothing \times \{2, 3\}$

(c) $(\mathbb{P}\,\varnothing) \times \{1\}$

(d) $\{(1, 2)\} \times \{3, 4\}$

□

Exercise 5.5 (Equality of sets) To show that two sets are equal, we must establish that they have exactly the same elements. A suitable proof will involve establishing an equivalence of the form

$$x \in a \Leftrightarrow x \in b$$

generalising the result, and applying the rule 'extension'.

Such an equivalence can often be established using a series of rewriting steps. For example, we may show that

$$\begin{aligned}
& x \in (a \cup b) \\
& \quad \Leftrightarrow x \in a \vee x \in b && \text{[union]} \\
& \quad \Leftrightarrow x \in b \vee x \in a && \text{[commutativity of } \vee\text{]} \\
& \quad \Leftrightarrow x \in (b \cup a) && \text{[union]}
\end{aligned}$$

and hence that $a \cup b = b \cup a$. Using this approach, show that

(a) $a \cap a = a$

(b) $a \cup a = a$

(c) $a \cap \varnothing = \varnothing$

(d) $a \cup \varnothing = a$

(e) $a \cap (b \setminus a) = \varnothing$

(f) $a \cup (b \setminus a) = a \cup b$

(g) $a \setminus (b \cup c) = (a \setminus b) \cap (a \setminus c)$

(h) $a \setminus (b \cap c) = (a \setminus b) \cup (a \setminus c)$

□

Exercise 5.6 (Characteristic tuple) If more than one variable is declared in the declaration part of a set comprehension, then the default term part is given by the *characteristic tuple* of the declaration: a typical element of the Cartesian product of the ranges. For example, the declaration

$$a : \mathbb{N};\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N}$$

gives rise to the characteristic tuple (a, s, t), an element of $\mathbb{N} \times \mathbb{P}\,\mathbb{N} \times \mathbb{P}\,\mathbb{N}$. If p is the set

$$\{\ a : \mathbb{N};\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \wedge a \notin t\ \}$$

then p is equal to

$$\{\ a : \mathbb{N};\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in s \wedge a \notin t \bullet (a, s, t)\ \}$$

and a typical element of p would be $(1, \{1, 2\}, \{2, 3\})$. With this understanding, give one element from each of the following sets:

(a) $\{\ a : \mathbb{N};\ b : \mathbb{N} \mid a > b\ \}$

(b) $\{\ s : \mathbb{P}\,\mathbb{N};\ a : \mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in (s \cap t)\ \}$

(c) $\{\ s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid s \cap t = \varnothing\ \}$

□