Equality and Definite Description

Exercise 4.1 (Rewriting) The rule 'eq-sub' allows us to replace an expression that appears to the right of an equality with the expression that appears to the left. An entirely similar rule would be:

$$\frac{s = t \quad p[s/x]}{p[t/x]} \ [\text{eq}-\text{sub}']$$

Show that this rule is a derived rule of our proof system. □

Exercise 4.2 (Universal one-point rule) The one-point rule for existential quantification states that once we have identified the quantified variable, we may eliminate it. Write down a suitable one-point rule for the universal quantifier, and show that it is a derived rule of our proof system. □

Exercise 4.3 (The one-point rule as an equivalence) Use the one-point rule for existential quantification as an equivalence

$$(\exists x : a \bullet x = t \land p) \ \Leftrightarrow \ (t \in a \land p[t/x])$$

provided that x is not free in t

to simplify each of the following statements:

(a) $\forall x : \mathbb{N} \bullet \exists y : \mathbb{N} \bullet x = y + 1$

(b) $\exists x, y : \mathbb{N} \bullet x + y = 4 \land p$

(c) $\exists x : \mathbb{N} \bullet (x = 1 \land p) \lor (x = 2 \land q)$

□

Exercise 4.4 (Generalisation) The universal quantifier is a generalised form of conjunction

$$(\forall x : \{1,2,3\} \bullet p) \;\Leftrightarrow\; p[1/x] \land p[2/x] \land p[3/x]$$

The existential quantifier is a generalised form of disjunction

$$(\exists x : \{1,2,3\} \bullet p) \;\Leftrightarrow\; p[1/x] \lor p[2/x] \lor p[3/x]$$

Is there a propositional connective that is generalised by the unique existential quantifier $\exists_1$? □

Exercise 4.5 (μ statements) Which of the following statements are provable in our mathematical language?

(a) $(\mu\, a : \mathbb{N} \mid a = a + a) \;=\; 0$

(b) $(\mu\, b : \mathbb{N} \mid b = b * b) \;=\; 1$

(c) $(\mu\, c : \mathbb{N} \mid c > c + c) \;=\; (\mu\, c : \mathbb{N} \mid c > c + c)$

(d) $(\mu\, d : \mathbb{N} \mid d = d \;\mathsf{div}\; d) \;=\; 1$

□

Exercise 4.6 (μ expressions) Suggest μ-expressions for each of the following:

(a) the height of the highest mountain;

(b) the difference between n and 100, where n is the largest multiple of 8 that is less than 100;

(c) the number of words in the second longest chapter of this book.

□