Exercises for Chapter 2

Propositional Logic

Exercise 2.1 (Truth tables) Construct a truth table for each of the following propositions:

(a) $p \wedge q \Rightarrow p$

(b) $(\neg p \Rightarrow p \wedge q) \Leftrightarrow p$

(c) $p \wedge (p \Rightarrow q) \Rightarrow q$

□

Exercise 2.2 (Tautologies) Decide whether each of the following statements is a tautology:

(a) $p \vee q \;\Leftrightarrow\; (\neg p \vee \neg q) \wedge q$

(b) $p \vee q \;\Leftrightarrow\; (\neg p \wedge \neg q) \vee q$

(c) $p \wedge \neg p \;\Rightarrow\; p$

(d) $p \vee \neg p \;\Rightarrow\; \neg p$

(e) $p \Rightarrow (q \Rightarrow p)$

(f) $(p \Rightarrow q) \Rightarrow p$

□

Exercise 2.3 (Exclusive or) Suppose that we define a new operator for our language of propositional logic, 'exclusive or', with the symbol $\triangledown$. We want $p \triangledown q$ to be true if exactly one of p and q is true. That is, this new operator differs from *inclusive or* only in the case that both components are true.

(a) Draw up a truth-table for $\veebar$

(b) Devise suitable introduction and elimination rules for this operator.

□

Exercise 2.4 (Conjunction and disjunction) By exhibiting a proof tree in each case, show that each of the following is a theorem of our natural deduction system:

(a) $(p \land (q \lor r)) \Leftrightarrow ((p \land q) \lor (p \land r))$

(b) $(p \lor (q \land r)) \Leftrightarrow ((p \lor q) \land (p \lor r))$

These equivalences tell us that conjunction and disjunction distribute through each other. □

Exercise 2.5 (Implication and negation) By exhibiting a proof tree in each case, show that each of the following is a theorem of our natural deduction system:

(a) $(p \Rightarrow q) \Rightarrow (\neg p \lor q)$

(b) $(p \Leftrightarrow q) \Leftrightarrow ((p \land q) \lor (\neg p \land \neg q))$

□

Exercise 2.6 (Using equivalences) If we have established that an equivalence is a theorem of our deductive system, then we may use it in proofs. For example, we may use the equivalences

$$p \lor (p \land q) \Leftrightarrow (p \lor p) \land (p \lor q) \tag{2.1}$$

$$p \lor p \Leftrightarrow p \tag{2.2}$$

to show that $p \lor (p \land q) \Leftrightarrow p$ is a theorem:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil p \lor (p \land q) \rceil^{[1]}}{(p \lor p) \land (p \lor q)}\,[\Leftrightarrow\ 2.1]}{p \land (p \lor q)}\,[\Leftrightarrow\ 2.2]}{p}\,[\land\text{-elim1}]}{p \lor (p \land q) \Rightarrow p}\,[\Rightarrow\text{-intro}^{[1]}] \qquad \dfrac{\dfrac{\lceil p \rceil^{[2]}}{p \lor (p \land q)}\,[\lor\text{-intro1}]}{p \Rightarrow p \lor (p \land q)}\,[\Rightarrow\text{-intro}^{[2]}]}{p \lor (p \land q) \Leftrightarrow p}\,[\Leftrightarrow\text{-intro}]$$

(a) The basic proof rules of our natural deduction system can be applied only to complete expressions; they cannot be used to rewrite components. Explain why rules based upon equivalences allow us to do exactly this, as in the proof above.

(b) By constructing a suitable counter-example, show that it is safe to replace one component expression with another *only when* the two expressions are equivalent. In particular, show that the implication $p \Rightarrow q$ is not enough to justify the replacement of component p with q.

□

Exercise 2.7 (Using transitivity) Implication and equivalence are transitive,

$$\frac{p \Rightarrow q \quad q \Rightarrow r}{p \Rightarrow r}\ [\Rightarrow\text{-}*] \qquad \frac{p \Leftrightarrow q \quad q \Leftrightarrow r}{p \Leftrightarrow r}\ [\Leftrightarrow\text{-}*]$$

We may use this fact in the presentation of proofs of implication and equivalence. Suppose that we have the following equivalences as theorems:

$$p \vee q \Leftrightarrow q \vee p \tag{2.3}$$

$$\neg\neg p \Leftrightarrow p \tag{2.4}$$

$$p \Rightarrow q \Leftrightarrow (\neg p \vee q) \tag{2.5}$$

We may prove that

$$(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$$

is also a theorem by exhibiting the following proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow q \Leftrightarrow (\neg p \vee q)}\ [2.5] \quad \dfrac{}{\neg p \vee q \Leftrightarrow q \vee \neg p}\ [2.3]}{(p \Rightarrow q) \Leftrightarrow (q \vee \neg p)}\ [\Leftrightarrow\text{-}*] \quad \dfrac{}{\neg\neg q \Leftrightarrow q}\ [2.4]}{(p \Rightarrow q) \Leftrightarrow (\neg\neg q \vee \neg p)}\ [\Leftrightarrow\text{-}*] \quad \dfrac{}{(\neg\neg q \vee \neg p) \Leftrightarrow (\neg q \Rightarrow \neg p)}\ [2.5]}{(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)}\ [\Leftrightarrow\text{-}*]$$

We may simplify the presentation of this argument:

$$
\begin{array}{llr}
(p \Rightarrow q) & & \\
\quad \Leftrightarrow & (\neg p \vee q) & [\Leftrightarrow\ 2.5] \\
\quad \Leftrightarrow & (q \vee \neg p) & [\Leftrightarrow\ 2.3] \\
\quad \Leftrightarrow & (\neg\neg q \vee \neg p) & [\Leftrightarrow\ 2.4] \\
\quad \Leftrightarrow & (\neg q \Rightarrow \neg p) & [\Leftrightarrow\ 2.5]
\end{array}
$$

Following this example, present simplified proofs of each of the following equivalences:

(a) $\neg(p \Leftrightarrow q) \Leftrightarrow ((p \vee q) \wedge \neg(p \wedge q))$

(b) $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$

(c) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$

(d) $(p \Rightarrow \neg p) \Leftrightarrow \neg p$

(e) $(\neg p \Rightarrow p) \Leftrightarrow p$

(f) $(r \Rightarrow (p \wedge q)) \Leftrightarrow ((r \Rightarrow p) \wedge (r \Rightarrow q))$

(g) $((p \vee q) \Rightarrow r) \Leftrightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r))$

(h) $((p \wedge q) \Rightarrow r) \Leftrightarrow ((p \Rightarrow r) \vee (q \Rightarrow r))$

(i) $((p \wedge q) \Leftrightarrow p) \Leftrightarrow (p \Rightarrow q)$

(j) $((p \vee q) \Leftrightarrow p) \Leftrightarrow (q \Rightarrow p)$

(k) $(p \Rightarrow (q \Rightarrow r)) \Leftrightarrow ((p \wedge q) \Rightarrow r)$

(l) $(q \Rightarrow (p \Rightarrow r)) \Leftrightarrow (p \Rightarrow (q \Rightarrow r))$

□