Sequences

Solution 9.1 (Supermarket) In a supermarket, there are m checkouts open.

(a) At any instant, the supermarket checkouts may be modelled as a number of queues. Using an axiomatic definition, and assuming the set P of people, define the set of all possible configurations of the supermarket checkouts.

$$
\begin{array}{l}
SMQ : \mathbb{P}(\mathrm{seq}(\mathrm{seq}\, P)) \\
\hline
SMQ = \{\, qq : \mathrm{seq}(\mathrm{seq}\, P) \mid \\
\qquad \#qq = m \land \\
\qquad \forall\, i, j : \mathrm{dom}\, qq \bullet \\
\qquad\qquad \forall\, k : \mathrm{dom}(qq\, i);\ l : \mathrm{dom}(qq\, j) \bullet \\
\qquad\qquad\qquad i \neq j \lor k \neq l \Rightarrow qq\, i\, k \neq qq\, j\, l \,\}
\end{array}
$$

(b) Define a function that models the arrival of someone at the supermarket checkouts. The function should take a person as argument and put them into the shortest queue.

$$
\begin{array}{l}
arrive : P \times SMQ \rightarrowtail SMQ \\
\hline
\forall\, p : P;\ smq, smq' : SMQ \bullet \\
\quad arrive(p, smq) = smq' \Leftrightarrow \\
\qquad \exists\, i : \mathrm{dom}\, smq \bullet \\
\qquad\qquad \#(smq\, i) = \min\{\, j : \mathrm{dom}\, smq \bullet \#(smq\, j) \,\} \land \\
\qquad\qquad smq' = smq \oplus \{ i \mapsto (smq\, i) \frown \langle p \rangle \}
\end{array}
$$

(c) Define another function which models a person changing queues. The function should take three arguments: the person, the queue being left,

and the queue being joined.

$$
\begin{array}{|l}
change : P \times \mathbb{N} \times \mathbb{N} \times SMQ \rightarrowtail SMQ \\
\hline
\forall\, p : P;\ old, new : \mathbb{N};\ smq, smq' : SMQ \bullet \\
\quad change\,(p, old, new, smq) = smq' \Leftrightarrow \\
\qquad smq' = smq \oplus \{old \mapsto (smq\ old) \upharpoonright (P \setminus \{p\}), \\
\qquad\qquad\qquad new \mapsto (smq\ new) \frown \langle p \rangle\}
\end{array}
$$

(d) Define another function which models the opening of an $(m+1)$th queue. Customers may join the new queue, but they do not otherwise change places. Modelling the opening using a relation allows us to capture the variation arising from the decisions of individual customers.

$$
\begin{array}{|l}
customers : SMQ \rightarrow \mathbb{P}\, P \\
\hline
\forall\, smq : SMQ \bullet \\
\quad customers\ smq = \bigcup \{\, i : \mathrm{dom}\, smq \bullet \mathrm{ran}(smq\ i) \,\}
\end{array}
$$

$$
\begin{array}{|l}
open : SMQ \rightarrow SMQ \times \mathrm{seq}\, P \\
\hline
\forall\, smq, smq' : SMQ;\ q : \mathrm{seq}\, P \bullet \\
\quad open\ smq = (smq', q) \Leftrightarrow \\
\qquad customers\ smq = (customers\ smq') \cup (\mathrm{ran}\, q) \wedge \\
\qquad \forall\, i : \mathrm{dom}\, smq \bullet smq'\ i \subseteq smq\ i
\end{array}
$$

□

Solution 9.2 (Esrever) The induction principle for sequences:

$$
\frac{P\,\langle\rangle \qquad \forall\, x : X;\ t : \mathrm{seq}\, X \bullet P\, t \Rightarrow P\,(\langle x \rangle \frown t)}{\forall\, s : \mathrm{seq}\, X \bullet P\, s}\ [\text{induction}]
$$

We may use this to show that the *reverse* function is an *involution*: applying it a second time returns us to the original sequence. However, if we have only the defining equations

$$
reverse\,\langle\rangle = \langle\rangle \qquad \text{(reverse.1)}
$$

$$
reverse(\langle x \rangle \frown s) = (reverse\, s) \frown \langle x \rangle \qquad \text{(reverse.2)}
$$

then the obvious property will not suffice as an inductive hypothesis; to complete the proof, we would need to know that

$$
reverse\,(s \frown t) = reverse\, t \frown reverse\, s
$$

That is, that *reverse* is anti-distributive. This property is strong enough to be proved by induction on the first argument; we define a property

$$
\begin{array}{|l}
\hline
[X] \\
Q\,_ : \mathbb{P}\,\mathrm{seq}\,X \\
\hline
\forall\, s : \mathrm{seq}\,X \bullet \\
\quad Q\,s \Leftrightarrow \forall\, t : \mathrm{seq}\,X \bullet \mathit{reverse}\,(s \frown t) \;=\; \mathit{reverse}\,t \frown \mathit{reverse}\,s \\
\hline
\end{array}
$$

and exhibit a suitable proof tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil \forall\, t : \mathrm{seq}\,X \bullet \mathit{reverse}\,(u \frown t) = \mathit{reverse}\,t \frown \mathit{reverse}\,u \rceil^{[3]} \quad \lceil t \in \mathrm{seq}\,X \rceil^{[4]}}{\mathit{reverse}\,(u \frown t) = \mathit{reverse}\,t \frown \mathit{reverse}\,u}\,[\forall\text{-elim}] \quad \lceil x \in X \rceil^{[1]} \quad \lceil u \in \mathrm{seq}\,X \rceil^{[2]}}{\mathit{reverse}\,((\langle x \rangle \frown u) \frown t) = (\mathit{reverse}\,t) \frown (\mathit{reverse}\,\langle x \rangle \frown u)}\,[\text{lemma2}]}{\forall\, t : \mathrm{seq}\,X \bullet \mathit{reverse}\,((\langle x \rangle \frown u) \frown t) = (\mathit{reverse}\,t) \frown (\mathit{reverse}\,\langle x \rangle \frown u)}\,[\forall\text{-intro}^{[4]}]}{(\forall\, t : \mathrm{seq}\,X \bullet \mathit{reverse}\,(u \frown t) = \mathit{reverse}\,t \frown \mathit{reverse}\,u) \Rightarrow \forall\, t : \mathrm{seq}\,X \bullet \mathit{reverse}\,((\langle x \rangle \frown u) \frown t) = (\mathit{reverse}\,t) \frown (\mathit{reverse}\,\langle x \rangle \frown u)}\,[\Rightarrow\text{-intro}^{[3]}]}{Q\,u \Rightarrow Q\,(\langle x \rangle \frown u)}\,[\text{gendef}]}{\forall\, x : X;\ \forall\, u : \mathrm{seq}\,X \bullet Q\,u \Rightarrow Q\,(\langle x \rangle \frown u)}\,[\forall\text{-intro}^{[1][2]}] \quad \dfrac{\dfrac{}{Q\,\langle\rangle}\,[\text{lemma 1}]}{\forall\, s : \mathrm{seq}\,X \bullet Q\,s}\,[\text{induction}]}{\forall\, s, t : \mathrm{seq}\,X \bullet \mathit{reverse}\,(s \frown t) \;=\; \mathit{reverse}\,t \frown \mathit{reverse}\,s}\,[\text{gendef}]
$$

The two lemmas are easily established by equational reasoning: the first by

$$
\begin{aligned}
&\mathit{reverse}\,(\langle\rangle \frown t) \\
&\quad = \mathit{reverse}\,t && [\text{cat.1}] \\
&\quad = (\mathit{reverse}\,t) \frown \langle\rangle && [\text{cat.1}]
\end{aligned}
$$

and the second by

$$
\mathit{reverse}\,((\langle x \rangle \frown u) \frown t)
$$

$$
\begin{aligned}
&= \mathit{reverse}\,(\langle x\rangle \frown (u \frown t)) && [\text{cat.2}]\\
&= \mathit{reverse}\,(u \frown t) \frown \langle x\rangle && [\text{reverse.2}]\\
&= (\mathit{reverse}\,t \frown \mathit{reverse}\,u) \frown \langle x\rangle \\
& \qquad\qquad [\mathit{reverse}\,(u \frown t) = (\mathit{reverse}\,t \frown \mathit{reverse}\,u)]\\
&= \mathit{reverse}\,t \frown (\mathit{reverse}\,u \frown \langle x\rangle) && [\text{cat.2}]\\
&= \mathit{reverse}\,t \frown \mathit{reverse}\,(\langle x\rangle \frown u) && [\text{reverse.2}]
\end{aligned}
$$

We are now ready to prove the universal property asked for in the question. We begin by defining a suitable set:

$$
\begin{array}{|l}
\hline
[X] \\
P_ : \mathbb{P}\,\mathrm{seq}\,X \\
\hline
\forall\, s : \mathrm{seq}\,X \bullet \\
\quad P\,s \Leftrightarrow \mathit{reverse}\,(\mathit{reverse}\,s) = s \\
\hline
\end{array}
$$

Now consider the following proof tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil \mathit{reverse}\,(\mathit{reverse}\,t) = t\rceil^{[3]}}{\lceil x \in X\rceil^{[1]} \quad \lceil t \in \mathrm{seq}\,X\rceil^{[2]}}\ [\text{lemma 2}]}{\mathit{reverse}\,(\mathit{reverse}\,t) = t \ \Rightarrow\ \mathit{reverse}\,(\mathit{reverse}\,(\langle x\rangle \frown t)) = \langle x\rangle \frown t}\ [\Rightarrow\text{-intro}^{[3]}]}{P\,t \Rightarrow P\,(\langle x\rangle \frown t)}\ [\text{gendef}]}{\forall\, x : X;\ \forall\, t : \mathrm{seq}\,X \bullet P\,t \Rightarrow P\,(\langle x\rangle \frown t)}\ [\forall\text{-intro}^{[1][2]}]}{\dfrac{P\,\langle\rangle}{\forall\, s : \mathrm{seq}\,X \bullet P\,s}\ [\text{induction}]}\ [\text{lemma 1}]}{\forall\, s : \mathrm{seq}\,X \bullet \mathit{reverse}\,(\mathit{reverse}\,s) = s}\ [\text{gendef}]
$$

The two lemmas may be proved by equational reasoning:

$$
\begin{aligned}
&\mathit{reverse}\,(\mathit{reverse}\,\langle\rangle)\\
&\quad = \mathit{reverse}\,\langle\rangle && [\text{reverse.1}]\\
&\quad = \langle\rangle && [\text{reverse.1}]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathit{reverse}\,(\mathit{reverse}\,\langle x\rangle \frown t)\\
&\quad = \mathit{reverse}\,((\mathit{reverse}\,t) \frown \langle x\rangle) && [\text{reverse.2}]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathit{reverse}\,(\langle x\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && \text{[previous result]}\\
&= \mathit{reverse}\,(\langle x\rangle \frown \langle\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && \text{[cat.1]}\\
&= ((\mathit{reverse}\,\langle\rangle) \frown \langle x\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && \text{[reverse.2]}\\
&= (\langle\rangle \frown \langle x\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && \text{[reverse.1]}\\
&= \langle x\rangle \frown \mathit{reverse}\,(\mathit{reverse}\,t) && \text{[cat.1]}\\
&= \langle x\rangle \frown t && [\mathit{reverse}\,(\mathit{reverse}\,t) = t]
\end{aligned}
$$

This completes the inductive proof: we may conclude that for any set X

$$\forall\, s : \mathrm{seq}\, X \bullet \mathit{reverse}\,(\mathit{reverse}\, s) = s$$

and hence that the reverse operator is an involution. □