Solutions for Chapter 8

Functions

Solution 8.1 (Properties of functions) The set of 9 functions:

$$\{\, \emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,1)\}, \\ \{(1,0)\}, \{(0,0),(1,1)\}, \{(0,1),(1,1)\}, \\ \{(1,0),(0,0)\}, \{(0,1),(1,0)\} \,\}$$

(a) The set of total functions:

$$\{\, \{(0,0),(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(0,0)\}, \{(0,1),(1,0)\} \,\}$$

(b) The set of functions which are neither injective nor surjective:

$$\{\, \{(0,1),(1,1)\}, \{(0,0),(1,0)\} \,\}$$

(c) The set of functions which are injective but not surjective:

$$\{\, \emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\} \,\}$$

(d) There are no functions (of this type) which are surjective but not injective

(e) The set of bijective functions:

$$\{\, \{(0,0),(1,1)\}, \{(0,1),(1,0)\} \,\}$$

□

Solution 8.2 (Overriding and restriction)

$$
\begin{aligned}
& x \mapsto y \in A \lhd (f \oplus g) \\
& \quad \Leftrightarrow x \in A \wedge x \mapsto y \in f \oplus g && [\lhd] \\
& \quad \Leftrightarrow x \in A \wedge x \mapsto y \in ((\operatorname{dom} g) \unlhd f) \cup g && [\oplus] \\
& \quad \Leftrightarrow x \in A \wedge ((x \mapsto y \in (\operatorname{dom} g) \unlhd f) \vee (x \mapsto y \in g)) && [\cup] \\
& \quad \Leftrightarrow x \in A \wedge ((x \notin \operatorname{dom} g \wedge x \mapsto y \in f) \vee (x \mapsto y \in g)) && [\unlhd] \\
& \quad \Leftrightarrow (x \in A \wedge x \notin \operatorname{dom} g \wedge x \mapsto y \in f) \vee (x \in A \wedge x \mapsto y \in g) && [\wedge] \\
& \quad \Leftrightarrow (x \mapsto y \in f \wedge (x \in A \wedge x \notin (A \cap \operatorname{dom} g))) && [\cap, \wedge] \\
& \qquad \vee \\
& \qquad (x \in A \wedge x \mapsto y \in g) \\
& \quad \Leftrightarrow x \mapsto y \in (A \cap \operatorname{dom} g) \unlhd (A \cap f) \cup (A \lhd g) && [\lhd] \\
& \quad \Leftrightarrow x \mapsto y \in ((A \lhd f) \oplus (A \lhd g)) && [\oplus]
\end{aligned}
$$

□

Solution 8.3 (Injectivity, image, and intersection)

$$
\begin{aligned}
& z \in f(\!| A_1 \cap A_2 |\!) \\
& \quad \Leftrightarrow z \in \{ x : X;\ y : Y \mid x \in (A_1 \cap A_2) \wedge x \mapsto y \in f \bullet y \} && [(\!|\ |\!)] \\
& \quad \Leftrightarrow z \in \{ x : X;\ y : Y \mid x \in A_1 \wedge x \in A_2 \wedge x \mapsto y \in f \bullet y \} && [\cap] \\
& \quad \Leftrightarrow z \in \{ x : X;\ y : Y \mid (x \in A_1 \wedge x \mapsto y \in f) && [\wedge] \\
& \qquad\qquad\qquad\qquad \wedge \\
& \qquad\qquad\qquad\qquad (x \in A_2 \wedge x \mapsto y \in f) \bullet y \} \\
& \quad \Leftrightarrow z \in \{ y : Y \mid (\exists x : X \bullet (x \in A_1 \wedge x \mapsto y \in f) \\
& \qquad\qquad\qquad\qquad\qquad \wedge \\
& \qquad\qquad\qquad\qquad\qquad (x \in A_2 \wedge x \mapsto y \in f)) \} \\
& \quad \Leftrightarrow z \in \{ y : Y \mid (\exists x : X \bullet (x \in A_1 \wedge x \mapsto y \in f)) && [\exists] \\
& \qquad\qquad\qquad \wedge \\
& \qquad\qquad\qquad (\exists x : X \bullet (x \in A_2 \wedge x \mapsto y \in f)) \} \\
& \quad \Leftrightarrow z \in \{ y : Y \mid \exists x : X \bullet x \in A_1 \wedge x \in A_2 \wedge x \mapsto y \in f \} && [\wedge] \\
& \quad \Leftrightarrow z \in f(\!| A_1 |\!) \cap f(\!| A_2 |\!) && [(\!|\ |\!)]
\end{aligned}
$$

□

Solution 8.4 (Approximation) This problem is not as easy as it appears: firstly, the notion of closest approximation is not necessarily functional; secondly, the resulting function or relation will not be total. The following definition introduces a function with the desired property, but does not explain what value it takes when there are two elements of A equidistant from i:

$$
\begin{array}{|l}
closest : \mathbb{Z} \rightarrow \mathbb{P}\,\mathbb{Z} \nrightarrow \mathbb{Z} \\
\hline
\forall\, i : \mathbb{Z};\ A : \mathbb{P}_1\,\mathbb{Z};\ c : \mathbb{Z} \bullet \\
\quad c = closest\ i\ A \in A \Leftrightarrow \\
\qquad c \in A \wedge \qquad \forall\, a : A \bullet |\, a - i \,| \geq |\, c - i \,|
\end{array}
$$

The declaration tells us that *closest* is a function, and thus that it has a unique value, even in the case

$$A = \{1, 3\} \qquad i = 2$$

but it does not always tell us what this value should be. As such, it is a loose definition, correctly reflecting the looseness in the original specification.

The following definition reflects a tighter specification, stating that if there are two values in A equidistant from i, then $closest\ i\ A$ is the lesser of the two:

$$
\begin{array}{|l}
closest : \mathbb{Z} \rightarrow \mathbb{P}\,\mathbb{Z} \nrightarrow \mathbb{Z} \\
\hline
\forall\, i : \mathbb{Z};\ A : \mathbb{P}_1\,\mathbb{Z};\ c : \mathbb{Z} \bullet \\
\quad c = closest\ i\ A \in A \Leftrightarrow \\
\qquad c \in A \wedge \qquad \forall\, a : A \mid a \neq c \bullet |\, a - i \,| \geq |\, c - i \,| \vee c < a
\end{array}
$$

Neither definition gives a value to the function when the set of available integers is empty. □