Solutions for Chapter 7

Relations

Solution 7.1 (Extension)

(a) $\{\,\emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\},$
$\{(1,0),(1,1)\}, \{(0,0),(0,1)\}, \{(0,1),(1,1)\},$
$\{(0,1),(1,0)\}, \{(0,0),(1,1)\}, \{(0,0),(1,0)\},$
$\{(0,0),(1,0),(1,1)\}, \{(0,0),(0,1),(1,1)\},$
$\{(0,0),(0,1),(1,0)\}, \{(0,1),(1,0),(1,1)\},$
$\{(0,0),(0,1),(1,0),(1,1)\}\,\}$

(b) $\{\,\emptyset, \{(0,0)\}, \{(0,1)\}, \{(0,0),(0,1)\}\,\}$

(c) $\{\,\emptyset\,\}$

(d) $\{\,\emptyset\,\}$

□

Solution 7.2 (Domain and range)

(a) $\{0,1,2\}$

(b) $\{1,2,3\}$

(c) $\{1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3\}$

(d) $\emptyset$

□

Solution 7.3 (Domain and range restriction)

(a)

$$
\begin{aligned}
& x \mapsto y \in A \lhd (B \lhd R) \\
& \quad \Leftrightarrow x \in A \land x \mapsto y \in (B \lhd R) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{domain restriction}] \\
& \quad \Leftrightarrow x \in A \land x \in B \land x \mapsto y \in R \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{domain restriction}] \\
& \quad \Leftrightarrow x \in A \cap B \land x \mapsto y \in R \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{intersection}] \\
& \quad \Leftrightarrow x \in (A \cap B) \lhd R \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{domain restriction}]
\end{aligned}
$$

(b)

$$
\begin{aligned}
& x \mapsto y \in (R \cup S) \rhd C \\
& \quad \Leftrightarrow x \mapsto y \in (R \cup S) \land y \in C \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{range restriction}] \\
& \quad \Leftrightarrow (x \mapsto y \in R \lor x \mapsto y \in S) \land y \in C \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{union}] \\
& \quad \Leftrightarrow (x \mapsto y \in R \land y \in C) \lor (x \mapsto y \in S \land y \in C) \\
& \qquad\qquad\qquad\qquad\qquad\qquad [\text{since } (P \lor Q) \land R \Leftrightarrow (P \land R) \lor (Q \land R)] \\
& \quad \Leftrightarrow x \mapsto y \in R \rhd C \lor x \mapsto y \in S \rhd C \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{range restriction}] \\
& \quad \Leftrightarrow x \mapsto y \in (R \rhd C) \cup (S \rhd C) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{union}]
\end{aligned}
$$

(c)

$$
\begin{aligned}
& x \mapsto y \in (A \setminus B) \lhd R \\
& \quad \Leftrightarrow x \in (A \setminus B) \land x \mapsto y \in R \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{domain restriction}] \\
& \quad \Leftrightarrow x \in A \land \neg (x \in B) \land x \mapsto y \in R \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{set minus}] \\
& \quad \Leftrightarrow x \in A \land x \mapsto y \in R \land \neg (x \in B) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\text{properties of } \land] \\
& \quad \Leftrightarrow x \in A \land x \mapsto y \in R \land (\neg (x \in B) \lor \neg (x \mapsto y \in R) \\
& \qquad\qquad\qquad\qquad\qquad\qquad [\text{since } P \land Q \Leftrightarrow P \land (Q \lor \neg P)] \\
& \quad \Leftrightarrow x \mapsto y \in (A \lhd R) \land \neg (x \in B \land x \mapsto y \in R)
\end{aligned}
$$

[De Morgan]

$$\Leftrightarrow x \mapsto y \in (A \lhd R) \wedge \neg\, (x \mapsto y \in B \lhd R)$$

[domain restriction]

$$\Leftrightarrow x \mapsto y \in (A \lhd R) \setminus (B \lhd R)$$

[set minus]

□

Solution 7.4 (Relatives)

(a) $parent == child^{\sim}$

(b) $sibling == (parent \fatsemi child) \setminus \mathrm{id}[Person]$

(c) $cousin == child \fatsemi sibling \fatsemi parent$

(d) $ancestor == parent^{+}$

□

Solution 7.5 (More relatives)

(a) $mother == Female \lhd parent$

(b) $grandmothers == \mathrm{dom}(mother \fatsemi parent)$

(c) $only_son == (\mathrm{dom}\, sibling) \lhd (Male \lhd child)$

(d) $kissing == (Male \lhd cousins \rhd Female)^{\mathrm{s}}$

□

Solution 7.6 (Composition)

(a) $\{2 \mapsto 4, 3 \mapsto 3, 3 \mapsto 4, 4 \mapsto 2\}$

(b) $\{1 \mapsto 3, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}$

(c) $\{1 \mapsto 1, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4\}$

(d) $\{1 \mapsto 4, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 1\}$

□

Solution 7.7 (Find the relations)

(a) R is $\{0 \mapsto 1, 1 \mapsto 0\}$

(b) S is either $\{0 \mapsto 0, 0 \mapsto 1, 1 \mapsto 1\}$ or $\{0 \mapsto 0, 1 \mapsto 0, 1 \mapsto 1\}$

(c) T is either $\{0 \mapsto 1\}$ or $\{1 \mapsto 0\}$

□

Solution 7.8 (Closures)

(a) $R^{\mathrm{r}} = \{ a, b : \mathbb{N} \mid b = a \lor b = a + 1 \}$

(b) $R^{\mathrm{s}} = \{ a, b : \mathbb{N} \mid |a - b| = 1 \}$

(c) $R^{+} = \{ a, b : \mathbb{N} \mid b > a \}$

(d) $R^{*} = \{ a, b : \mathbb{N} \mid b \geq a \}$

(e) $S^{\mathrm{r}} = \{ x, y : \mathbb{Z} \mid y = x \lor y = |x| \}$

(f) $S^{\mathrm{s}} = \{ x, y : \mathbb{Z} \mid |y| = |x| \}$

(g) $S^{+} = S$

(h) $S^{*} = S^{\mathrm{r}}$

□

Solution 7.9 (Transitivity and subset) We begin with a chain of equivalences:

$$
\begin{aligned}
& S \in \{ R : X \leftrightarrow X \mid \forall\, x, y, z : X \bullet x \mapsto y \in R \land y \mapsto z \in R \Rightarrow x \mapsto z \in R \} \\
& \quad \Leftrightarrow \forall\, x, y, z : X \bullet x \mapsto y \in S \land y \mapsto z \in S \Rightarrow x \mapsto z \in S \\
& \qquad\qquad\qquad\