Sets

Solution 5.1 (Membership)

(a) $1 \in \{4, 3, 2, 1\}$ is true.

(b) $\{1\} \in \{1, 2, 3, 4\}$ is undefined.

(c) $\{1\} \in \{\{1\}, \{2\}, \{3\}, \{4\}\}$ is true.

(d) $\varnothing \in \{1, 2, 3, 4\}$ is undefined.

□

Solution 5.2 (Comprehension) Given that $*$ denotes multiplication, that div denotes integer division, and mod denotes remainder, describe each of the following using set comprehension:

(a) $\{ z : \mathbb{Z} \mid 0 \leq z \leq 100 \}$

(b) $\{ z : \mathbb{Z} \bullet 10 * z \}$

(c) $\{ z : \mathbb{Z} \bullet z \mathsf{ mod } 2 = 0 \land z \mathsf{ mod } 3 = 0 \land z \mathsf{ mod } 5 = 0 \}$

□

Solution 5.3 (Power sets)

(a) $\mathbb{P}\{1, 2\}$ is the set $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$

(b) $\mathbb{P}\,\mathbb{P}\{1,2\}$ is the set

$$\begin{aligned}&\{\varnothing, \{\varnothing\}, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\},\\ &\ \{\varnothing, \{1\}\}, \{\varnothing, \{2\}\}, \{\varnothing, \{1,2\}\},\\ &\ \{\{1\}, \{2\}\}, \{\{1\}, \{1,2\}\}, \{\{2\}, \{1,2\}\},\\ &\ \{\varnothing, \{1\}, \{2\}\}, \{\varnothing, \{1\}, \{1,2\}\},\\ &\ \{\varnothing, \{2\}, \{1,2\}\}, \{\{1\}, \{2\}, \{1,2\}\}, \{\varnothing, \{1\}, \{2\}, \{1,2\}\}\}\end{aligned}$$

(c) $\mathbb{P}\,\varnothing$ is the set $\{\varnothing\}$

(d) $\mathbb{P}\,\mathbb{P}\,\varnothing$ is the set $\{\varnothing, \{\varnothing\}\}$

□

Solution 5.4 (Cartesian products)

(a) $\{1\} \times \{2,3\}$ is the set $\{(1,2), (1,3)\}$

(b) $\varnothing \times \{2,3\}$ is the set $\varnothing$

(c) $(\mathbb{P}\,\varnothing) \times \{1\}$ is the set $\{(\varnothing, 1)\}$

(d) $\{(1,2)\} \times \{3,4\}$ is the set $\{((1,2),3), ((1,2),4)\}$

□

Solution 5.5 (Equality of sets)

(a)

$$\begin{aligned}&x \in a \cap a\\ &\quad \Leftrightarrow (x \in a \wedge x \in a) && \text{[inter]}\\ &\quad \Leftrightarrow (x \in a) && \text{[idempotence of } \wedge\text{]}\\ &\quad \Leftrightarrow x \in a\end{aligned}$$

(b)

$$\begin{aligned}&x \in a \cup a\\ &\quad \Leftrightarrow (x \in a \vee x \in a) && \text{[union]}\\ &\quad \Leftrightarrow (x \in a) && \text{[idempotence of } \vee\text{]}\\ &\quad \Leftrightarrow x \in a\end{aligned}$$

(c)

$$
\begin{aligned}
& x \in a \cap \varnothing \\
& \quad \Leftrightarrow x \in a \wedge x \in \varnothing && \text{[inter]} \\
& \quad \Leftrightarrow x \in a \wedge \mathit{false} && \text{[property of } \varnothing\text{]} \\
& \quad \Leftrightarrow \mathit{false} && \text{[property of } \mathit{false}\text{]} \\
& \quad \Leftrightarrow x \in \varnothing && \text{[property of } \varnothing\text{]}
\end{aligned}
$$

(d)

$$
\begin{aligned}
& x \in a \cup \varnothing \\
& \quad \Leftrightarrow x \in a \vee x \in \varnothing && \text{[inter]} \\
& \quad \Leftrightarrow x \in a \vee \mathit{false} && \text{[property of } \varnothing\text{]} \\
& \quad \Leftrightarrow x \in a && \text{[property of } \mathit{false}\text{]}
\end{aligned}
$$

(e)

$$
\begin{aligned}
& x \in a \cap (b \setminus a) \\
& \quad \Leftrightarrow x \in a \wedge x \in (b \setminus a) && \text{[inter]} \\
& \quad \Leftrightarrow x \in a \wedge (x \in b \wedge x \notin a) && \text{[set minus]} \\
& \quad \Leftrightarrow x \in a \wedge (x \notin a \wedge x \in b) && \text{[commutativity]} \\
& \quad \Leftrightarrow (x \in a \wedge x \notin a) \wedge x \in b && \text{[associativity]} \\
& \quad \Leftrightarrow \mathit{false} \wedge x \in b && \text{[property of } \mathit{false}\text{]} \\
& \quad \Leftrightarrow \mathit{false} && \text{[property of } \mathit{false}\text{]} \\
& \quad \Leftrightarrow x \in \varnothing && \text{[property of } \varnothing\text{]}
\end{aligned}
$$

□

Solution 5.6 (Characteristic tuple)

(a) $\{\, a : \mathbb{N};\ b : \mathbb{N} \mid a > b \,\}$ has as an element $(1, 0)$

(b) $\{\, s : \mathbb{P}\,\mathbb{N};\ a : \mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid a \in (s \cap t) \,\}$ has as an element $(\{1, 2\}, 1, \{1, 2, 3\})$

(c) $\{\, s : \mathbb{P}\,\mathbb{N};\ t : \mathbb{P}\,\mathbb{N} \mid s \cap t = \varnothing \,\}$ has as an element $(\{1, 2\}, \{3\})$

□