Solutions for Chapter 4

Equality and Definite Description

Solution 4.1 (Rewriting)

$$\dfrac{\dfrac{s = t}{t = s}\,[\text{eq}-\text{symm}] \qquad p[s/x]}{p[t/x]}\,[\text{eq}-\text{sub}]$$

□

Solution 4.2 (Universal one-point rule) A suitable rule would be

$$\dfrac{\forall\, x : a \bullet x = t \Rightarrow p}{t \in a \Rightarrow p[t/x]} \qquad \text{provided that } x \text{ is not free in } t$$

We will prove the rule in both directions. In the forward direction, the proof relies upon the reflexivity of equality:

$$\dfrac{\dfrac{\dfrac{}{t = t}\,[\text{eq}-\text{ref}] \qquad \dfrac{[t \in a]^{[1]} \quad \forall\, x : a \bullet x = t \Rightarrow p}{t = t \Rightarrow p[t/x]}\,[\forall-\text{elim}]}{p[t/x]}\,[\Rightarrow-\text{elim}]}{t \in a \Rightarrow p[t/x]}\,[\Rightarrow-\text{intro}^{[1]}]$$

In the other direction, the proof relies upon Leibniz's principle:

$$\dfrac{\dfrac{\dfrac{\lceil x = t\rceil^{[2]} \qquad \dfrac{\dfrac{\lceil x = t\rceil^{[2]} \quad \lceil x \in a\rceil^{[1]}}{t \in a}\,[\text{eq}-\text{sub}']}{p[t/x]} \qquad t \in a \Rightarrow p[t/x]}{p}\,[\text{eq}-\text{sub}]}{x = t \Rightarrow p}\,[\Rightarrow-\text{intro}^{[2]}]}{\forall\, x : a \bullet x = t \Rightarrow p}\,[\forall-\text{intro}^{[1]}]\,[\Rightarrow-\text{elim}]$$

□

Solution 4.3 (The one-point rule and arithmetic) Recall that *simplify* means to rewrite in a syntactically simpler, but semantically equivalent, form.

(a)

$$\begin{aligned}
&\forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet x = y + 1 \\
&\quad \Leftrightarrow \forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet y = x - 1 && \text{[arithmetic]} \\
&\quad \Leftrightarrow \forall\, x : \mathbb{N} \bullet x - 1 \in \mathbb{N} && \text{[one-point rule]} \\
&\quad \Leftrightarrow \mathit{false} && \text{[defns. of } - \text{ and } \mathbb{N}]
\end{aligned}$$

The final equivalence holds because $0 \in \mathbb{N}$ and yet $0 - 1 \notin \mathbb{N}$. We may assume that the subtraction operator is defined for all integers.

(b)

$$\begin{aligned}
&\exists\, x, y : \mathbb{N} \bullet x + y = 4 \wedge p \\
&\quad \Leftrightarrow \exists\, x, y : \mathbb{N} \bullet y = 4 - x && \text{[arithmetic]} \\
&\quad \Leftrightarrow \exists\, x : \mathbb{N} \bullet 4 - x \in \mathbb{N} && \text{[one-point rule]} \\
&\quad \Leftrightarrow \mathit{true}
\end{aligned}$$

The final equivalence holds because $0 \in \mathbb{N}$ and $4 - 0 = 4 \in \mathbb{N}$.

(c)

$$\begin{aligned}
&\exists\, x : \mathbb{N} \bullet (x = 1 \wedge p) \vee (x = 2 \wedge q) \\
&\quad \Leftrightarrow (\exists\, x : \mathbb{N} \bullet x = 1 \wedge p) \vee (\exists\, x : \mathbb{N} \bullet x = 2 \wedge q) && \text{[law of } \exists] \\
&\quad \Leftrightarrow (1 \in \mathbb{N} \wedge p[1/x]) \vee (\exists\, x : \mathbb{N} \bullet x = 2 \wedge q) && \text{[one-point rule]} \\
&\quad \Leftrightarrow (1 \in \mathbb{N} \wedge p[1/x]) \vee (2 \in \mathbb{N} \wedge q[2/x]) && \text{[one-point rule]} \\
&\quad \Leftrightarrow p[1/x] \vee q[2/x] && \text{[defn. of } \mathbb{N}]
\end{aligned}$$

□

Solution 4.4 (Generalisation) Semantically, the unique existential quantifier is a generalised form of the exclusive-or operator $\veebar$. The statement

$$\exists_1 x : a \bullet p$$

is true if property p is true of *exactly one* element of a; the statement

$$p[a_1/x] \veebar p[a_2/x]$$

is true if property p is true of *exactly one* of a_1 and a_2.

However, the reader should not be tempted to extend this relationship to the syntactic domain; the following statement is *not* true

$$(\exists_1 x : \{1, 2, 3\} \bullet p) \;\Leftrightarrow\; p[1/x] \veebar (p[2/x] \veebar p[3/x])$$

□

Solution 4.5 (μ statements)

(a) $(\mu\, a : \mathbb{N} \mid a = a + a) = 0$ is a provable statement, since 0 is the only natural number with the specified property.

(b) $(\mu\, b : \mathbb{N} \mid b = b * b) = 1$ is not provable. The specified property is true of both 0 and 1, and thus the value of the *mu* expression is undefined.

(c) $(\mu\, c : \mathbb{N} \mid c > c + c) = (\mu\, c : \mathbb{N} \mid c > c + c)$ is a provable statement. Neither expression is properly defined, but we may conclude that they are equal; there is little else that we can prove about them.

(d) $(\mu\, d : \mathbb{N} \mid d = d \;\mathsf{div}\; d) = 1$ is not a provable statement. We cannot confirm that 1 is the only natural number with the specified property; we do not know what value is taken by $0 \;\mathsf{div}\; 0$.

□

Solution 4.6 (μ expressions)

(a) $(\mu\, m : \mathit{Mountain} \mid$
$\quad (\forall\, n : \mathit{Mountain} \setminus \{m\} \bullet$
$\quad\quad \mathit{height}(n) < \mathit{height}(m)) \bullet \mathit{height}(m))$

(b) $(\mu\, n : \mathbb{N} \mid n = \max\{\, m : \mathbb{N} \mid 8 * m < 100 \bullet 8 * m \,\} \bullet 100 - n)$

(c) $(\mu\, c : \mathit{Chapter} \mid (\exists_1\, d : \mathit{Chapter} \bullet \mathit{length}(d) > \mathit{length}(c)) \bullet \mathit{length}(c))$

□