Solutions for Chapter 2

Propositional Logic

Solution 2.1 (Truth tables)

(a)

p	$\wedge$	q	$\Rightarrow$	p
f	f	f	t	f
f	f	t	t	f
t	f	f	t	t
t	t	t	t	t

(b)

$(\neg p$	$\Rightarrow$	p	$\wedge$	$q)$	$\Leftrightarrow$	p
t	f	f	f	f	t	f
t	f	f	f	t	t	f
f	t	t	f	f	t	t
f	t	t	t	t	t	t

(c)

p	$\wedge$	$(p$	$\Rightarrow$	$q)$	$\Rightarrow$	q
f	f	f	t	f	t	f
f	f	f	t	t	t	t
t	f	t	f	f	t	f
t	t	t	t	t	t	t

□

Solution 2.2 (Tautologies)

(a) $p \vee q \;\Leftrightarrow\; (\neg p \vee \neg q) \wedge q$ is a contingency

(b) $p \vee q \;\Leftrightarrow\; (\neg p \wedge \neg q) \vee q$ is a contingency

(c) $p \wedge \neg p \;\Rightarrow\; p$ is not

(d) $p \vee \neg p \;\Rightarrow\; \neg p$ is not

(e) $p \Rightarrow (q \Rightarrow p)$ is a tautology

(f) $(p \Rightarrow q) \Rightarrow p$ is not

□

Solution 2.3 (Exclusive or)

(a) The exclusive-or operator admits the following truth table:

p	q	$p \veebar q$
f	f	f
f	t	t
t	f	t
t	t	f

(b) A suitable introduction rule for this operator might be

$$\frac{(p \wedge \neg q) \vee (\neg p \wedge q)}{p \veebar q}\ [\veebar\text{-intro}]$$

while any of the following could serve as elimination rules

$$\frac{p \mathbin{\overline{\vee}} q}{p \vee q}\ [\overline{\vee}\text{-elim1}]$$

$$\frac{p \mathbin{\overline{\vee}} q}{\neg p \vee \neg q}\ [\overline{\vee}\text{-elim2}]$$

$$\frac{p \mathbin{\overline{\vee}} q}{(\neg p \wedge q) \vee (p \wedge \neg q)}\ [\overline{\vee}\text{-elim3}]$$

□

Solution 2.4 (Conjunction and disjunction)

(a) In one direction:

$$\dfrac{\dfrac{\lceil (p \wedge q) \vee (p \wedge r) \rceil^{[3]} \quad \dfrac{\dfrac{\dfrac{\lceil p \wedge q \rceil^{[4]}}{p}\ [\wedge\text{-elim1}] \quad \dfrac{\dfrac{\lceil p \wedge q \rceil^{[4]}}{q}\ [\wedge\text{-elim2}]}{q \vee r}\ [\vee\text{-intro1}]}{p \wedge (q \vee r)}\ [\wedge\text{-intro}] \quad \dfrac{\dfrac{\dfrac{\lceil p \wedge r \rceil^{[4]}}{p}\ [\wedge\text{-elim1}] \quad \dfrac{\dfrac{\lceil p \wedge r \rceil^{[4]}}{r}\ [\wedge\text{-elim2}]}{q \vee r}\ [\vee\text{-intro2}]}{p \wedge (q \vee r)}\ [\wedge\text{-intro}]}{p \wedge (q \vee r)}\ [\vee\text{-elim}^{[4]}]}{((p \wedge q) \vee (p \wedge r)) \Rightarrow (p \wedge (q \vee r))}\ [\Rightarrow\text{-intro}^{[3]}]$$

and the other

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil p \wedge (q \vee r)\rceil^{[1]}}{p}\,[\wedge\text{−elim1}] \qquad \lceil r\rceil^{[2]}}{p \wedge r}\,[\wedge\text{−intro}]}{(p \wedge q) \vee (p \wedge r)}\,[\vee\text{−intro2}]}{\dfrac{\dfrac{\dfrac{\lceil p \wedge (q \vee r)\rceil^{[1]}}{p}\,[\wedge\text{−elim1}] \qquad \lceil q\rceil^{[2]}}{p \wedge q}\,[\wedge\text{−intro}]}{(p \wedge q) \vee (p \wedge r)}\,[\vee\text{−intro1}]}}{\dfrac{\dfrac{\lceil p \wedge (q \vee r)\rceil^{[1]}}{q \vee r}\,[\wedge\text{−elim2}]}{(p \wedge q) \vee (p \wedge r)}\,[\vee\text{−elim}^{[2]}]}}{(p \wedge (q \vee r)) \Rightarrow ((p \wedge q) \vee (p \wedge r))}\,[\Rightarrow\text{−intro}^{[1]}]
$$

We may then combine these two proofs with $\Leftrightarrow$−*intro*.

□

Solution 2.5 (Implication and negation)

(a) In one direction

$$
\dfrac{\dfrac{\dfrac{}{p \vee \neg p}\,[\text{excluded middle}] \qquad \dfrac{\dfrac{\lceil p \Rightarrow q\rceil^{[1]} \quad \lceil p\rceil^{[2]}}{q}\,[\Rightarrow\text{−elim}]}{\neg p \vee q}\,[\vee\text{−intro2}]}{(\neg p \vee q)}\,[\vee\text{−elim}^{[2]}]}{(p \Rightarrow q) \Rightarrow (\neg p \vee q)}\,[\Rightarrow\text{−intro}^{[1]}]
$$

and the other

$$\dfrac{\dfrac{\dfrac{[\neg p \vee q]^{[3]} \qquad \dfrac{\dfrac{[p]^{[4]} \quad [\neg p]^{[5]}}{false}\,[\neg\text{–elim}]}{q}\,[\text{false–elim}] \qquad [q]^{[5]}}{q}\,[\vee\text{–elim}^{[5]}]}{p \Rightarrow q}\,[\Rightarrow\text{–intro}^{[4]}]}{(\neg p \vee q) \Rightarrow (p \Rightarrow q)}\,[\Rightarrow\text{–intro}^{[3]}]$$

We may then combine these two proofs with $\Leftrightarrow$–*intro*.

□

Solution 2.6 (Using equivalences)

(a) The value of a compound proposition is determined entirely by the values of its components. If $q \Leftrightarrow r$ then q and r have the same value, and we can safely replace q with r in a compound proposition expression such as $p \wedge q$ without changing the value of $p \wedge q$.

(b) Suppose that we wished to rewrite the proposition

$$p \Leftrightarrow (p \wedge p)$$

using the (valid) implication $p \Rightarrow (p \vee q)$. If we replace the first instance of p with $p \vee q$, then we obtain the statement

$$(p \vee q) \Leftrightarrow (p \wedge p)$$

which is true only if it is also the case that $q \Rightarrow p$. Thus an implication alone is not enough to justify the replacement of one component with another.

□

Solution 2.7 (Using transitivity)

(a)

$$
\begin{aligned}
&\neg(p \Leftrightarrow q) \\
&\quad \Leftrightarrow \neg((p \Rightarrow q) \wedge (q \Rightarrow p)) && [\Leftrightarrow \text{ and } \Rightarrow] \\
&\quad \Leftrightarrow \neg(p \Rightarrow q) \vee \neg(q \Rightarrow p) && [\text{De Morgan}] \\
&\quad \Leftrightarrow \neg(\neg p \vee q) \vee \neg(q \Rightarrow p) && [\Leftrightarrow \text{ and } \vee] \\
&\quad \Leftrightarrow \neg(\neg p \vee q) \vee \neg(\neg q \vee p) && [\Leftrightarrow \text{ and } \vee] \\
&\quad \Leftrightarrow \neg(\neg p \vee q) \vee (\neg\neg q \wedge \neg p) && [\text{De Morgan}] \\
&\quad \Leftrightarrow (\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg p) && [\text{De Morgan}] \\
&\quad \Leftrightarrow (p \wedge \neg q) \vee (\neg\neg q \wedge \neg p) && [\text{idempotence of } \neg] \\
&\quad \Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p) && [\text{idempotence of } \neg] \\
&\quad \Leftrightarrow (p \vee (q \wedge \neg p)) \wedge (\neg q \vee (q \wedge \neg p)) && [\text{distribution}] \\
&\quad \Leftrightarrow (p \vee q) \wedge (p \vee \neg p) \wedge (\neg q \vee q) \wedge (\neg q \vee \neg p) && [\text{distribution}] \\
&\quad \Leftrightarrow (p \vee q) \wedge \mathit{true} \wedge (\neg q \vee q) \wedge (\neg q \vee \neg p) && [\text{excluded middle}] \\
&\quad \Leftrightarrow (p \vee q) \wedge (\neg q \vee q) \wedge (\neg q \vee \neg p) && [\wedge \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow (p \vee q) \wedge \mathit{true} \wedge (\neg q \vee \neg p) && [\text{excluded middle}] \\
&\quad \Leftrightarrow (p \vee q) \wedge (\neg q \vee \neg p) && [\wedge \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow (p \vee q) \wedge \neg(q \wedge p) && [\text{De Morgan}] \\
&\quad \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q) && [\text{commutativity}]
\end{aligned}
$$

□