Predicate logic

Exercise 3.1 (Defining predicates) Define a pair of predicates p and q involving free variable x such that

(a) $\forall\, x : \mathbb{N} \bullet p$ is false but $\exists\, x : \mathbb{N} \bullet p$ is true.

(b) $\exists\, x : \mathbb{N} \bullet p \Rightarrow q$ is true but $\forall\, x : \mathbb{N} \bullet p \Rightarrow q$ is false.

□

Exercise 3.2 (Some propositions) The predicate p with free variables x and y is defined by

$$p \;\Leftrightarrow\; x + y > x^2$$

Which of the following are true?

(a) $\forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet p$

(b) $\exists\, y : \mathbb{N} \bullet \forall\, x : \mathbb{N} \bullet p$

(c) $(\forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet p) \Rightarrow (\exists\, y : \mathbb{N} \bullet \forall\, x : \mathbb{N} \bullet p)$

(d) $(\exists\, y : \mathbb{N} \bullet \forall\, x : \mathbb{N} \bullet p) \Rightarrow (\forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet p)$

(e) $(\forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet p) \Leftrightarrow (\exists\, y : \mathbb{N} \bullet \forall\, x : \mathbb{N} \bullet p)$

□

Exercise 3.3 (Distribution) We have proved that the universal quantifier distributes through conjunction in one direction:

$$(\forall\, x : a \bullet p \wedge q) \Rightarrow (\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q)$$

Demonstrate that it distributes in the other direction by providing a proof of

$$(\forall\, x : a \bullet p) \wedge (\forall\, x : a \bullet q) \Rightarrow (\forall\, x : a \bullet p \wedge q)$$

□

Exercise 3.4 (Theorems) By exhibiting a proof tree in each case, show that the following statements are all theorems of our deductive system (that is, construct a proof tree ending in this statement that has no undischarged assumptions and no hypotheses):

(a) $(\neg\, \forall\, x : a \bullet p) \Leftrightarrow (\exists\, x : a \bullet \neg p)$

(b) $(\neg\, \exists\, x : a \bullet p) \Leftrightarrow (\forall\, x : a \bullet \neg p)$

(c) $(\exists\, x : a \bullet p \vee q) \Leftrightarrow ((\exists\, x : a \bullet p) \vee (\exists\, x : a \bullet q))$

(d) $(\exists\, x : a \bullet p \Rightarrow q) \Leftrightarrow ((\forall\, x : a \bullet p) \Rightarrow (\exists\, x : a \bullet q))$

(e) $(\forall\, x : a \bullet p \Rightarrow q) \Rightarrow ((\forall\, x : a \bullet p) \Rightarrow (\forall\, x : a \bullet q))$

(f) $(\exists\, x : a \bullet p \wedge q) \Rightarrow ((\exists\, x : a \bullet p) \wedge (\exists\, x : a \bullet q))$

□

Exercise 3.5 (Problem) What is wrong with the following ‘proof’?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil \forall\, x : a \bullet \exists\, y : b \bullet p \rceil^{[1]} \quad \lceil x \in a \rceil^{[2]}}{\exists\, y : b \bullet p}\,[\forall\text{-elim}] \qquad \lceil p \rceil^{[3]}}{p}\,[\exists\text{-elim}^{[3]}]}{\forall\, x : a \bullet p}\,[\forall\text{-intro}^{[2]}]}{\exists\, y : b \bullet \forall\, x : a \bullet p}\,[\exists\text{-intro}]}{(\forall\, x : a \bullet \exists\, y : b \bullet p) \Rightarrow (\exists\, y : b \bullet \forall\, x : a \bullet p)}\,[\Rightarrow\text{-intro}^{[1]}]$$

□

Exercise 3.6 (Another problem) The following predicate is *not* a theorem of our deductive system:

$$(\forall\, x : a \bullet p \vee q) \Rightarrow ((\forall\, x : a \bullet p) \vee (\forall\, x : a \bullet q))$$

For at least one choice of a, p, and q, this predicate is false: suppose that a were the natural numbers $\mathbb{N}$, with p as ‘x is even’ and q as ‘x is odd’. Hence the

following derivation must be either invalid or incomplete:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\forall x : a \bullet p \vee q]^{[1]}}{p \vee q}\,[\forall\text{–elim}] \qquad \dfrac{\dfrac{[p]^{[2]}}{(\forall x : a \bullet p)}\,[\forall\text{–intro}]}{(\forall x : a \bullet p) \vee (\forall x : a \bullet q)}\,[\vee\text{–intro1}] \qquad \dfrac{\dfrac{[q]^{[2]}}{(\forall x : a \bullet q)}\,[\forall\text{–intro}]}{(\forall x : a \bullet p) \vee (\forall x : a \bullet q)}\,[\vee\text{–intro2}]}{(\forall x : a \bullet p) \vee (\forall x : a \bullet q)}\,[\vee\text{–elim}^{[2]}]}{(\forall x : a \bullet p \vee q) \Rightarrow (\forall x : a \bullet p) \vee (\forall x : a \bullet q)}\,[\Rightarrow\text{–intro}^{[1]}]
$$

Identify the step or steps in the proof tree that are unjustified. □