Optimal Control of Transient Behavior in Coupled Solid State Lasers

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Abstract

We show that optimal control theory can be applied to substantially reduce transient times for transitions between in-phase and out-of-phase states in coupled solid state lasers. The control is a time-varying optical field that is injected into the cavities of each laser. Numerical simulations indicate that the transient times can be reduced by an order of magnitude when increasing briefly the injection strength by a factor of three. PACS: 05.45.Xt; 42.55.-f; 42.60.Da; 42.60.Fc

Laser arrays are promising for space communication applications. Such applications require compact sources with high optical intensities and fast switching times [1]. Recently, both solid state [2, 3] and semiconductor [4, 5] laser arrays have been investigated, and various relevant aspects of their dynamical behavior such as chaotic synchronization [6], chaotic communication [7], and amplitude drop-out [8] have already been reported.

The most efficient mode of operation for space communication is realized when the array elements are synchronized to an in-phase (IP) state, such that the output interferes constructively and the light intensity at the central lobe scales as N^2, where N is the number of lasers in the array. Unfortunately, the IP state is typically unstable; instead, the system is driven to the stable out-of-phase (OP) state, whose destructive interference pattern results in low output intensities at the central lobe [3, 5]. The IP behavior can be stabilized by injecting a common driving laser field into the laser array elements [3, 5, 9]. For sufficiently high driving amplitude, the elements are entrained and the output intensities interfere constructively; full entrainment of the array is realized above a certain threshold, determined by the coupling of the array elements [3, 5].

Besides synchronization, an equally important issue in applications is the time required to reach the IP behavior from an arbitrary state. In particular, it is desirable to minimize the transient time between the IP and OP states (upon removal of the injected entrainment field). This aspects is important, for example, in fast switching and communications applications.

Despite the obvious practical relevance and potential of the topics, transient behavior, switching, and control thereof in (arrays of) lasers have not been widely studied. Porta et. al. [10] used a two-step steering function applied to the pumping of a single CO_2 laser to reduce the turn-on time by a factor of three. Using chaos control methods, Uchida et al [11] reporteded statistical properites of the transient response times between periodic attractors. Switching was realized via high frequency injection in a laser diode subject to optical feedback. Lippi et al [12] developed a global steering (targeting) technique to induce transitions between states of generic 2D separable nonlinear systems described by a Lotka-Volterra model.

In this Letter, we apply optimal control (OC) to reduce the transient switching times between OP and IP states in an array of two coupled solid state lasers. To the best of our knowledge, no control technique - optimal or otherwise - has been applied to reduce transient times in arrays of lasers. We target the approach at a system of coupled solid state (and, possibly, semiconducotr) lasers and demonstrate its efficiency. The OC method is completely general and systematic and does not depend essentially on internal features of the system, such as characteristic times. Since it tailors precisely the effort to the desired task, the OC method keeps the cost at its possible minimum and yields reductions of the transient time by a factor of ten, without resulting in overshoots.

OC of systems of ordinary differential equations was developed by Pontryagin and his co-workers in the 1950's [13]. The basic idea is to adjust a coefficient or a source term (taken as the control) in the differential system to maximize (or minimize) a goal that is represented in terms of the control and corresponding solution (state) of the differential system. This theory has been applied to a wide variety of applications, ranging from economics and management to physical and biological models [14, 15] and

was extended to partial differential equations by Lions [16]. The Pontryagn's Maximum Principle for OC of ordinary differential equations does not carry over directly to partial differential equations, but some of the associated techniques do [17]. For the sake of simplicity, we present the application of OC to a system of two coupled lasers, but the procedure extends canonically to arbitrary arrays.

For many dynamical regimes of of the laser array the intensities and gain oscillations are not large. Then the the complete description of the system can be reduced to the "phase model" [3, 8]:

$$\begin{aligned} \dot{\phi_1}(t) &= \delta_1 + \kappa sin(\phi_2 - \phi_1) - A_e sin\phi_1 \\ \dot{\phi_2}(t) &= \delta_2 + \kappa sin(\phi_1 - \phi_2) - A_e sin\phi_2. \end{aligned} \tag{1}$$

where $\delta_i, i = 1, 2$ are the detunings, κ is the coupling constant, and A_e is the amplitude of the injected field. Ideally, to entrain an array of N identical lasers requires an injected field amplitude $A_e = 4|\kappa|$. By solving the complete system, we verified numerically that we operate in a regime where the phase model gives an appropriate description of the dynamics.

We consider the physically relevant transitions between OP and IP states. We distinguish the two cases by their initial/final conditions and by a different labeling of the time- dependent entrainment amplitudes, $A_o(t)$ and $A_i(t)$, respectively.

1. For the case of initial OP lasers, taking the injected field amplitude to be a function of time, $A_o(t)$, the initial conditions read:

$$0 < \phi_1(0) < \pi \text{ and } \phi_2(0) = \phi_1(0) - \pi. \tag{2}$$

The control function, $A_o(t)$, is a Lebesgue integrable function such that

$$4|\kappa| \leq A_o(t) \leq M_o. \tag{3}$$

The lower bound is set at $4|\kappa|$ since $4|\kappa|$ is the constant amplitude external field input that will drive the phases to the final IP state, and M_o is an arbitrary upper bound.

Since the aim is to drive the phases, ϕ_1 and ϕ_2, close together over a certain time interval, $[0, T]$, we consider the following objective functional, which is to be minimized as a function of A_o:

$$J_o(A_o) = \frac{1}{2}\int_0^T [(\phi_1 - \phi_2)^2(t) + \varepsilon A_o^2(t)]dt. \tag{4}$$

The term εA_o^2 is a stabilizing term that represents "the cost of the control"; the positive parameter ε is chosen small to make the $(\phi_1 - \phi_2)^2$ term dominant. We seek the optimal input control $A_o^\star(t)$ such that

$$J_o(A_o^\star) = min_{4|\kappa| \leq A_o \leq M_o} J_o(A_o).$$

The existence of an OC, A_o^*, and corresponding optimal state pair, ϕ_1^*, ϕ_2^*, is guaranteed by the convexity of the problem as a function of the control and by the Lipschitz property of the right hand side of (1) in ϕ_1 and ϕ_2 [18].

This problem is solved using Pontryagin's Maximum Principle [13, 15], which converts the problem (1) - (3) into a problem of minimizing pointwise a Hamiltonian, H,

$$\begin{aligned} H = \frac{1}{2}(\phi_1 - \phi_2)^2 + \frac{\varepsilon}{2}A_o^2 + \lambda_{o1}(\delta_1 + \kappa sin(\phi_2 - \phi_1) - A_o(t)sin\phi_1) \\ +\lambda_{o2}(\delta_2 + \kappa sin(\phi_1 - \phi_2) - A_o(t)sin\phi_2), \end{aligned} \tag{5}$$

with respect to A_o. The adjoint functions, λ_{o1} and λ_{o2}, act like "Lagrange multipliers" and couple the differential equations (System 1) to the minimization problem. Pontryagin's Maximum Principle gives the system of differential equations and boundary conditions that the adjoint functions, λ_{o1} and λ_{o2} satisfy

$$\begin{aligned} \dot{\lambda}_{o1} &= -\partial H/\partial\phi_1 \\ &= -\phi_1 + \phi_2 + \lambda_{o1}(\kappa cos(\phi_2 - \phi_1) + A_o(t)cos\phi_1) - \lambda_{o2}\kappa cos(\phi_1 - \phi_2) \\ \dot{\lambda}_{o2} &= -\partial H/\partial\phi_2 \\ &= \phi_1 - \phi_2 - \lambda_{o1}\kappa cos(\phi_2 - \phi_1) + \lambda_{o2}(\kappa cos(\phi_1 - \phi_2) + A_o(t)cos\phi_2) \end{aligned} \tag{6}$$

$$\lambda_{o1}(T) = 0, \lambda_{o2}(T) = 0 \text{ (transversality conditions).} \tag{7}$$

To minimize the Hamiltonian with respect to A_o, we use the necessary condition $\partial H/\partial A_o = 0$ at $A_o^\star$, and solve for $A_o^\star$, taking the bounds into account. As a result, we obtain an explicit characterization of the OC as a function of the state and adjoint functions:

$$A_o^\star(t) = min(max[4|\kappa|, \frac{1}{\varepsilon}(\lambda_{o1}(t)sin\phi_1(t) + \lambda_{o2}(t)sin\phi_2(t))], M_o). \tag{8}$$

Note that concavity of H with respect to A_o yields $\partial^2 H/\partial A_o^2 = \varepsilon > 0$, which ensures we are indeed finding a minimizer.

The OC $A_o^\star$ is found by solving the optimality system (OS), i.e. the phase equations (1), (2) and the adjoint equations (6), (7) together with the explicit characterization (Eq. 8) of the OC.

2. For the case of initial IP lasers, the initial conditions read:

$$\frac{\pi}{2} < \phi_1(0) < \frac{3\pi}{2} \text{ and } \phi_1(0) = \phi_2(0). \tag{9}$$

To distinguish it from the previous case, we denote the control (injected field) by $A_i(t)$ (instead of $A_o(t)$) and the upper bound by M_i (instead of M_o) and require:

$$0 \leq A_i(t) \leq M_i.$$

Here the lower bound 0 for the controls was chosen since even with zero external field the system would be driven to the *out-of-phase* case. We seek now to minimize the objective functional

$$J_i(A_i) = \frac{1}{2}\int_0^T [(\pi - (\phi_1 - \phi_2))^2(t) + \varepsilon A_i^2(t)]dt \tag{10}$$

in order to drive the lasers to the OP case. This functional will drive the system such that $\phi_2(t)$ is close to $\phi_1(t) + \pi$.

A similar control argument as before yields

$$\begin{aligned} \dot{\lambda_{i1}} &= \pi - \phi_1 + \phi_2 + \lambda_{i1}(\kappa cos(\phi_2 - \phi_1) + A_i(t)cos\phi_1) - \lambda_{i2}\kappa cos(\phi_1 - \phi_2) \\ \dot{\lambda_{i2}} &= -(\pi - \phi_1 + \phi_2) \\ &- \lambda_{i1}\kappa cos(\phi_2 - \phi_1) + \lambda_{i2}(\kappa cos(\phi_1 - \phi_2) + A_i(t)cos\phi_2) \end{aligned} \tag{11}$$

$$\lambda_{i1}(T) = 0, \lambda_{i2}(T) = 0 \text{ (transversality conditions) .} \tag{12}$$

and

$$A_i^\star(t) = min(max[0, \frac{1}{\varepsilon}(\lambda_{i1}(t)sin\phi_1(t) + \lambda_{i2}(t)sin\phi_2(t))], M_i). \tag{13}$$

To find the explicit OC, we again solve the corresponding OS (1), (9), (11), and (12), using the characterization (13). Now we discuss the numerical solutions of the two OS for various choices of M_o and M_i. Since the state system has initial conditions and the adjoint system has final time conditions, the OS cannot be solved by an ordinary forward marching scheme. Instead, an iterative method with a fourth order Runge-Kutta scheme is used whereby the iterative method consists of the following steps for this two-point boundary value problem, namely:

(i) Guess the value of the OC ($A_i^\star$ or $A_o^\star$) over the simulated time; (ii) Solve the state system *forward* in time (for a time period $0 \leq t \leq T$) using the Runge-Kutta scheme; (iii) Solve the adjoint system *backward* in time for a time period ($T \geq t \geq 0$) using the Runge-Kutta scheme and the solution of the state equations from step (ii) (start with final time conditions (Eq. (12)) and step backwards); (iv) Update the control by using a convex combination of the previous control and the value calculated from the characterization (8) or (13) by using the new values of the states and adjoint functions; and (v) Repeat steps (i)-(iv) until the difference of values of unknowns at the present iteration and the previous iteration becomes arbitrarily small.

We present the numerical results for the following two cases: (i) Initial OP case: $\phi_1(0) = \frac{\pi}{2}$ and $\phi_2(0) = -\frac{\pi}{2}$; and (ii) Initial IP: $\phi_1(0) = \phi_2(0) = \pi$. The physical and numerical parameters are given in the figure captions. Since our goal is to reduce the transition time from the initial OP to the final IP case and vice-versa and it takes around $t = 2$ to go through a transient period for both of cases using a constant input ($A_o \equiv 4|\kappa|$ or $A_i \equiv 0$), all simulations run until $t = 2$. These two cases with time-varying OC have been investigated and compared with the case with the constant control case in the following respect:

- Total output intensity (in the units of the intensity of a single uncoupled laser) [8],

$$I_{total}/I_0 = 4cos^2[(\phi_1 - \phi_2)/2]$$

  where I_0 is the single laser intensity.

- OC as a function of time, $A^\star(t)$ (an injected field amplitude).

Various upper bounds, M_o and M_i, have been used.

In Figure 1, we present the results when laser are initially at the OP state. The value of the injection amplitude A_o to obtain the IP behavior (for the parameters of our choice) is given by $A_o = 4|\kappa| = 4$ (since in our simulations $\kappa = -1$). We use the OC to decrease the transient time to reach the IP behavior. We have chosen five equally-spaced upper bounds between 4 and 14. In the top frame, the corresponding total output intensities (solid lines) for different OC values and the total intensity without the OC (using a constant input $A_o = 4$) (dotted line) are plotted. As expected, the transient time shortens as controls are allowed to take higher values. In the bottom frame, five optimal amplitudes of the injected field during the simulated time, corresponding to the five different upper bounds, are displayed. Note that for each upper bound there is an OC, A_o^*.

In Figure 2, we present the results when the lasers are initially IP, and the final state is the OP solution (without injection). Here the goal is to reduce the transient time to reach the OP behavior. Five equally-spaced upper bounds are chosen between 0 and 10. In the top frame, the corresponding total output intensities (solid lines) and total intensity using without the injected field (dotted line) are plotted. Again, the transient time when using an OC is shorter then the one recorded for a constant control ($A_i = 0$). In the bottom frame, five optimal amplitudes of the injected field during the simulated time, corresponding to the five different upper bounds are displayed.

These results demonstrate that OC theory is a very efficient systematic tool for significantly reducing the transient times for the array of coupled lasers when switched between the IP and OP states. It is important to mention that the "price" for significant reduction of transient times is increasing briefly the injection strength. In our example, an increase by a factor of three in injection may lead to a decrease of the transient time by an order of magnitude. Since the typical injection power required to mode-lock single laser is less than 1% of the total output power of the injected laser, the "price" for such transient reduction in switching application is reasonable and worthwhile.

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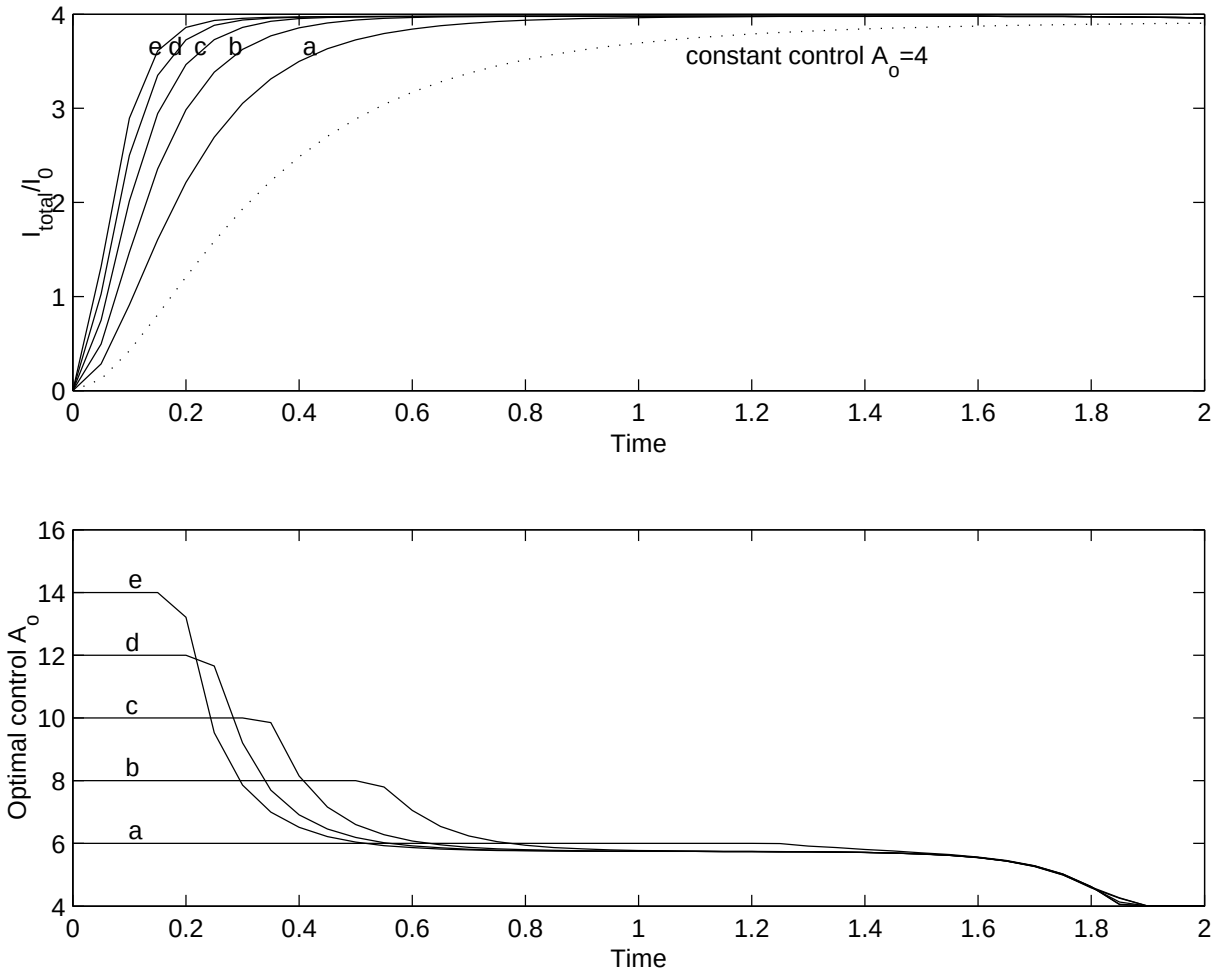

Figure 1: In the top frame, the total output intensities (solid lines) for different OC values and the total intensity without the OC (using a constant set $A_o = 4|\kappa| = 4$) (dotted line) are plotted. In the bottom frame, the corresponding optimal amplitudes of the injected field, A_o, during the simulated time, corresponding to the five different upper bounds are displayed. The parameters are: final time $T = 2$, time step $dt = 0.05$, stabilizing constant $\epsilon = 0.001$, upper bound $M_o = 4, 6, 8, 10, 12, 14$, detuning $\delta_1 = 0.2818$, detuning $\delta_2 = -0.268$, evanescent coupling constant $\kappa = -1$.

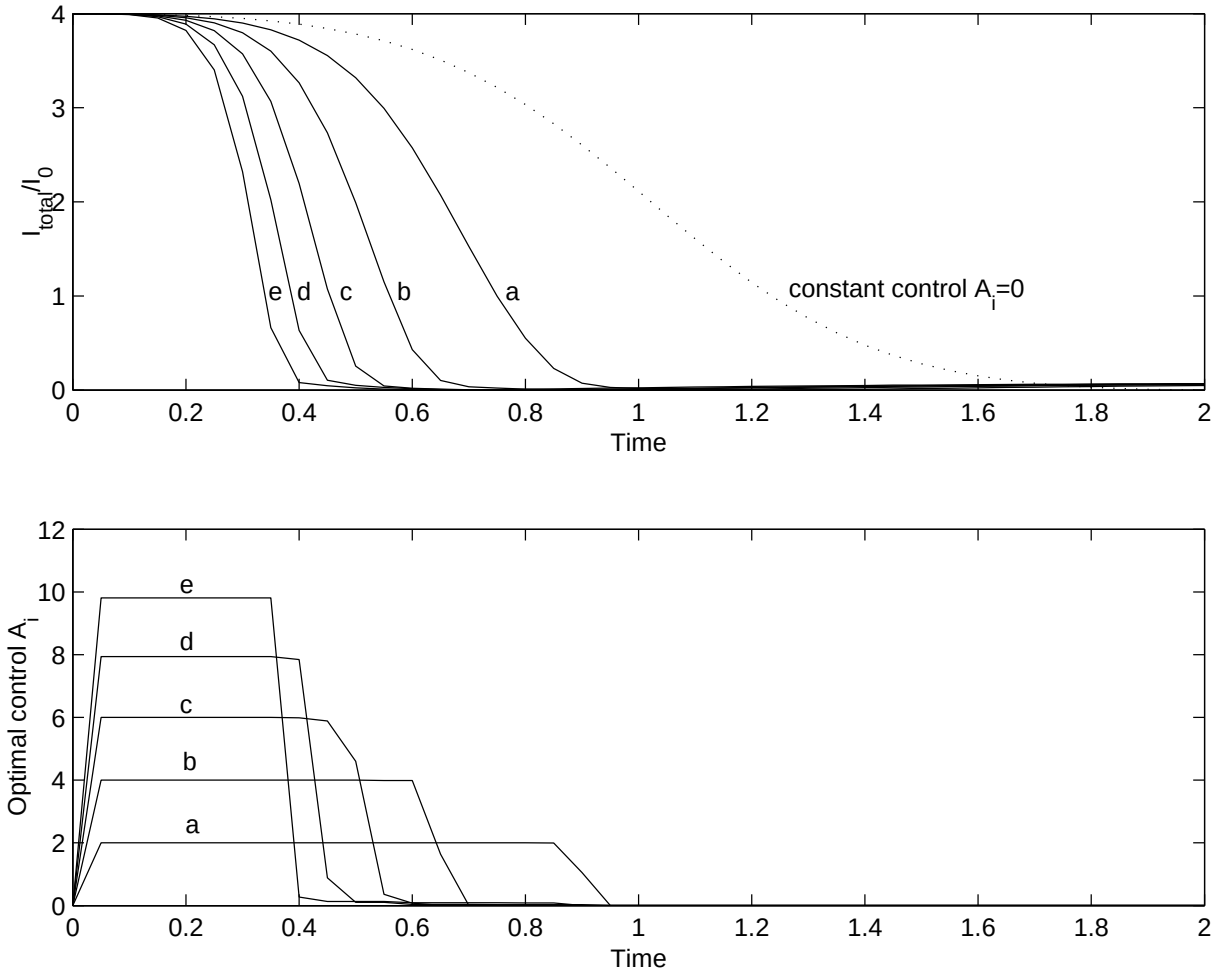

Figure 2: In the top frame, the total output intensities for different OC (solid lines) and the total intensity without the injected field (dotted line) are plotted. In the bottom frame, the corresponding optimal amplitudes of the injected field, A_i, during the simulated time, corresponding to the five different upper bounds ($M_i = 0, 2, 4, 6, 8, 10$), are displayed. The other parameters are the same as in Figure 1.

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