A FASTER GALERKIN BOUNDARY INTEGRAL ALGORITHM

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Abstract

The symmetry present in Green's functions is exploited to significantly reduce the matrix assembly time for a Galerkin boundary integral analysis. A relatively simple modification of the standard Galerkin implementation for computing the nonsingular integrals yields a twenty to thirty per cent decrease in computation time. This faster Galerkin method is developed for both singular and hypersingular equations, and applied to Symmetric-Galerkin implementations in two-dimensions for the Laplace equation and for orthotropic elasticity. In three dimensions, the modified algorithm has been implemented for the singular equation for the Laplace and elastodynamics equations. Comparison timing results for standard and modified algorithms are presented.

1 Introduction

The Galerkin boundary integral approximation [2, 12] has several important advantages over the more widely used collocation method. Foremost is the ability to evaluate, without ambiguity, hypersingular integrals [14] using standard continuous ele-

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ments [6, 8]. With collocation, existence of these integrals depends upon a (numerically difficult) differentiability constraint [7, 15, 18, 19], though recent work indicates that, in computations, this can possibly be relaxed [5, 20, 21]. Other benefits of Galerkin include higher accuracy, a simple and reliable analysis at corners, and a well developed mathematical theory [24]. Moreover, with Galerkin it is possible to formulate a natural *symmetric* boundary integral method [2]. However, these advantages come by way of a second boundary integration, and thus the overriding disadvantage of Galerkin has been significantly longer computation time, roughly an order of magnitude more than collocation. Understandably, this has severely limited the use of Galerkin in engineering applications.

The Symmetric-Galerkin boundary integral algorithm, first introduced by Sirtori [25] and Hartmann *et al.* [11], and subsequently developed by Maier and co-workers [17, 26] and others, has partially removed the dramatic performance gap between collocation and Galerkin [1]. In this approach, a symmetric coefficient matrix is obtained by utilizing Galerkin together with an appropriate combination of singular and hypersingular equations. The symmetry of the matrix can then be exploited to reduce the Gaussian elimination operations from $M^3/3$ to $M^3/6$, where M is the size of the linear system. Thus, for sufficiently large scale problems, significant savings are possible.

For M large, a direct solution algorithm would probably be replaced by an iterative scheme [13, 27]. The symmetry can still be exploited in the iterative alogorithm to reduce the effort by approximately one-half, but as the computation time should be much less than a direct matrix factorization, the savings over a non-symmetric formulation is far less than $M^3/6$. For this reason, and just in general, it would clearly be desirable to further reduce Galerkin computation time by taking full advantage of symmetry during the matrix construction phase of the calculation. The most natural approach is to evaluate only the upper (or lower) triangle, skipping integrations wherever possible. This strategy is partially effective for a few specific applications, *e.g.*, crack [4] or interface problems [16], but in general it has been difficult to exploit symmetry during matrix assembly. The reasons for this will be discussed in Section 2.

The purpose of this paper is to present an alternative implementation procedure for the Galerkin method, and to demonstrate that it results a much faster algorithm, roughly a $20 - 30\%$ reduction in computation time. Rather than skipping certain integrations, this approach in effect calculates *all* matrix elements, but in a more efficient way than a straightforward Galerkin algorithm. This algorithm appears to work well for both scalar and vector problems, and for simple and complicated Green's

functions. When combined with the other advantages of Galerkin, the Symmetric-Galerkin approach becomes an even more attractive alternative to collocation for engineering applications.

2 Faster Galerkin

2.1 Standard Galerkin

To present the new algorithm, the Galerkin and Symmetric-Galerkin procedures are very briefly described, in the context of the Laplace equation, $\nabla^2\phi = 0$. Further details on implementating Galerkin can be found in [26]. For Laplace, the boundary integral equations are for potential ϕ and normal flux $\partial\phi/\partial\mathbf{n} = \nabla\phi\bullet\mathbf{n}$ and are given by

$$\mathcal{P}(P) \equiv \phi(P) + \int_\Gamma \phi(Q)\frac{\partial G}{\partial \mathbf{n}}(P,Q)\,dQ - \int_\Gamma G(P,Q)\frac{\partial \phi}{\partial \mathbf{n}}\,dQ = 0 \ , \tag{1}$$

$$\mathcal{F}(P) \equiv \frac{\partial \phi(P)}{\partial \mathbf{N}} + \int_\Gamma \phi(Q)\frac{\partial^2 G}{\partial \mathbf{N}\partial \mathbf{n}}(P,Q)\,dQ - \int_\Gamma \frac{\partial G}{\partial \mathbf{N}}(P,Q)\frac{\partial \phi}{\partial \mathbf{n}}(Q)\,dQ = 0 \ . \tag{2}$$

Here $\mathbf{n} = \mathbf{n}(Q)$, $\mathbf{N} = \mathbf{N}(P)$ denote the unit outward normal on the boundary Γ of the domain $\mathcal{D}$, P and Q are points on Γ and $G(P,Q)$ denotes the Green's function. The important symmetry properties of G and its derivatives are

$$\begin{aligned} G(P,Q) &= G(Q,P) \\ \frac{\partial G}{\partial \mathbf{n}}(P,Q) &= -\frac{\partial G}{\partial \mathbf{N}}(P,Q) = \frac{\partial G}{\partial \mathbf{N}}(Q,P) \\ \frac{\partial^2 G}{\partial \mathbf{n}\partial \mathbf{N}}(P,Q) &= \frac{\partial^2 G}{\partial \mathbf{n}\partial \mathbf{N}}(Q,P) \end{aligned} \tag{3}$$

In a weighted residual formulation the integral equations are satisfied 'on average' in the form

$$\int_\Gamma \psi_k(P)\mathcal{P}(P)\,dP = 0 \ , \tag{4}$$

$$\int_\Gamma \psi_l(P)\mathcal{F}(P)\,dP = 0 \ , \tag{5}$$

where $\psi_j(P)$ are selected weight functions. In a Galerkin scheme, these weights are the shape functions employed in interpolating the boundary functions ϕ and its normal

derivative. With Eq. (3), it can be shown that symmetry of the coefficient matrix for a general mixed boundary value problem results from enforcing Eq. (4) on the Dirichlet surface (ϕ specified) and Eq. (5) on the Neumann ($\partial\phi/\partial\mathbf{n}$ specified) surface [11, 25].

To understand the difficulties involved in trying to exploit symmetry during matrix construction, for either a Galerkin or a Symmetric-Galerkin implementation, consider the Symmetric-Galerkin procedure for a Dirichlet problem. In this case Eq. (4) is enforced everywhere on the boundary. Thus, the coefficient matrix comes from the integral

$$\oint_\Gamma \psi_m(P) \oint_\Gamma \frac{\partial\phi}{\partial\mathbf{n}}(Q) G(P,Q)\, dQ\ dP\ , \tag{6}$$

and symmetry follows from $G(P,Q) = G(Q,P)$. Computing this term requires an integration for every pair of elements $\{E_P,\ E_Q\}$, and it would be desirable to skip all calculations destined for the lower triangle of the matrix. However, the evaluation of this term has many quantities in common (*e.g.*, Gauss points, distance vector $r = \|Q - P\|$, shape functions, etc.) with the contribution to the right hand side

$$\oint_\Gamma \psi_m(P) \oint_\Gamma \phi(Q) \frac{\partial G}{\partial\mathbf{n}}(P,Q)\, dQ\ dP\ , \tag{7}$$

and consequently these integrals are calculated together. However, as $\mathbf{n} = \mathbf{n}(Q)$, this kernel function is not symmetric and all integrations must be carried out to evaluate Eq. (7). As a consequence, bypassing integrations must contend with the fact that:

1. Most of the 'overhead' required to compute Eq. (6) is still carried out in computing the right hand side contribution, Eq. (7);
2. Logical testing is required to see if a particular integral can be avoided;
3. In two dimensions there is a natural consecutive node ordering. In three dimensions however, the efficiency will depend upon the ordering of the boundary nodes.

For the two classes of problems mentioned in the introduction, those containing fractures or interfaces, the symmetry can be exploited because there is no right hand side computation for these special surfaces. In an interface problem [16], both potential and flux are unknown along the interface, whereas for a crack, the usual boundary condition is zero flux (or traction in elasticity). Thus, no right hand side terms occur in integrating over these surfaces, and this permits skipping part of the calculation. The savings in this case depend upon the relative size of the interface or crack area, but typical configurations yielded about a 15% reduction in computation time [10].

2.2 Modified Galerkin

To describe the new algorithm, we focus on the potential equation, Eq. (4). The procedure for Eq. (5) is analogous. The most direct way to implement the Galerkin outer P integration and inner Q integration is to calculate the integrals:

$$
\begin{aligned}
&\texttt{DO} \quad E_P = 1, N_E \\
&\quad \texttt{DO} \quad E_Q = 1, N_E \\
&\qquad \int_{E_P} \psi_m(P) \int_{E_Q} \psi_l(Q)\, G(P,Q)\, dQ\ dP \\
&\qquad \int_{E_P} \psi_m(P) \int_{E_Q} \psi_l(Q)\, \frac{\partial G}{\partial \mathbf{n}}(P,Q)\, dQ\ dP\ .
\end{aligned}
$$

Here N_E denotes the number of elements, and the boundary functions $\phi(Q)$ and $\partial\phi/\partial\mathbf{n}(Q)$ have been replaced by their approximations,

$$
\begin{aligned}
\phi(Q) &= \sum_l \phi(Q_l)\psi_l(Q) \\
\frac{\partial \phi}{\partial \mathbf{n}}(Q) &= \sum_l \frac{\partial \phi}{\partial \mathbf{n}}(Q_l)\psi_l(Q)\ ,
\end{aligned}
\tag{8}
$$

where the boundary nodes $\{Q_l\}$ comprise the element E_Q. The proposed modification is to simply do the $\{E_Q,\ E_P\}$ integration at the same time as $\{E_P,\ E_Q\}$:

$$
\begin{aligned}
&\texttt{DO} \quad E_P = 1, N_E \\
&\quad \texttt{DO} \quad E_Q = E_P + 1, N_E \\
&\int_{E_P} \psi_m(P) \int_{E_Q} \psi_l(Q)\, G(P,Q)\, dQ\ dP \leftrightarrow \int_{E_Q} \int_{E_P} \\
&\qquad \int_{E_P} \psi_m(P) \int_{E_Q} \psi_l(Q) \frac{\partial G}{\partial \mathbf{n}}(P,Q)\, dQ\ dP\ . \\
&\qquad \int_{E_Q} \psi_l(Q) \int_{E_P} \psi_m(P) \frac{\partial G}{\partial \mathbf{N}}(Q,P)\, dP\ dQ\ .
\end{aligned}
$$

This approach takes advantage of the available symmetry, in two ways. First, the flux integral, involving $G(P,Q)$, is not recomputed. All matrix elements derived from this integral are filled in (so testing to see if a specific integration can be skipped is not required), but the quadrature is only done once instead of twice. Note that for the hypersingular equation, the symmetric term is the potential integral involving the

hypersingular kernel.

In addition to the symmetric term, additional computational savings result from evaluating both potential integrals at the same time: almost all of the 'overhead' computations for the reversed pair $\{E_Q,\ E_P\}$ have been computed in treating $\{E_P,\ E_Q\}$. Note that the main change in reversing the elements is that the Green's function differentiation is with respect to different normals, $\mathbf{n} = \mathbf{n}(Q)$ for $\{E_P,\ E_Q\}$ and $\mathbf{N} = \mathbf{N}(P)$ for the reversed pair. So that both terms can be calculated at the same time, the main implementation task is therefore to delay introducing the normal into the calculation as long as possible. This is especially simple for a linear element, as the normals are constant over an element. Thus, the integrals of the gradient ∇G can be computed and the appropriate normal derivative then obtained by an inner product. For a higher order interpolation, the normal is a function of Q or P and must therefore be included in the inner integration. Nevertheless, as indicated above, computing these two integrals together saves repeating some overhead computations. For the hypersingular equation, the non-symmetric integral is once again the one involving a single derivative of G.

The calculations reported below will only consider the *non-singular* integrations. The *coincident* singular integration $E_P = E_Q$ will always be done separately (hence the modified inner integration loop begins at $E_P + 1$), and symmetry cannot be exploited for this calculation. However, there is also the separate *adjacent* singular integrations, and one could contemplate applying the modified method (both pairs of elements at the same time) to these singular calculations. The relative cost of the adjacent singular integrations, compared to the non-singular, can vary depending upon the complexity of the Green's function and the chosen quadrature rules. These calculations are expected to be a more significant factor for three dimensions. When the adjacent singular integration is expensive, applying this technique to these evaluations is definitely worth pursuing.

3 Timing Results

In this section, timing results for the 'direct' Galerkin implementation are compared to equivalent calculations using the modified algorithm presented in the previous section. Times for the non-singular integrations alone, as well as the complete algorithm (which includes the solution of the matrix equation), are reported. For this purpose, the particular geometries and the specific boundary conditions are irrelevant; examining the computational savings as a function of scalar vs. vector, size of the problem,

and complexity of the Green's function will be of interest. The calculations will use a linear element in two dimensions, a three-node linear triangle in three dimensions.

All reported results are averages of multiple runs of the programs in an attempt to minimize the effect of any background operations on the timing data.

3.1 Two dimensions

The first set of calculations are for the scalar Laplace equation and the vector orthotropic elasticity. These calculations employ a Symmetric-Galerkin approximation, and for the Laplace equation the performance of Eq. (4) and Eq. (5) are examined separately by solving a Dirichlet and a Neumann problem. Table 1 presents computation times, for the Laplace equation, for the Dirichlet problem, Table 2 displays the analogous results for a Neumann problem. Results for three discretizations, $M = 150,\ 300,\ 400$, M being the number of nodes, are reported.

Nodes	Routine	Original Time (s)	Modified Time (s)	Percent Reduction
150	nonsingular	2.24	1.44	35.7
	total	2.25	1.45	35.6
300	nonsingular	8.89	5.70	35.9
	total	8.94	5.72	36.0
450	nonsingular	19.98	12.71	36.4
	total	20.71	13.54	34.6

Table 1: A comparison of timing results for the original and modified Symmetric-Galerkin boundary integral algorithm for the two dimensional Laplace equation. Shown are the times for computing the nonsingular integrals in Eq. (4), and for the complete solution algorithm.

Note first that for these small to moderate sized problems, there is little change in the per cent reduction when the time for the singular integrations and the linear algebra solution is included. Second, the modified method is more effective for the singular equation than for the hypersingular, producing roughly a 35% savings compared to 20%. This is due to the fact that $G(P, Q)$, required by Eq. (4), requires a relatively expensive logarithm evaluation, $\log(r)$, whereas the derivative kernels are essentially rational functions of r. This indicates, not surprisingly, that the benefits of the modified method increase as the kernel evaluation becomes more expensive.

Table 3 shows timing results for a mixed boundary value problem for orthotropic

Nodes	Routine	Original Time (s)	Modified Time (s)	Percent Reduction
150	nonsingular	1.19	0.93	21.8
	total	1.41	1.13	19.9
300	nonsingular	4.73	3.60	23.9
	total	5.74	4.52	21.3
450	nonsingular	10.54	8.02	23.9
	total	13.52	10.80	20.1

Table 2: Timing results, as in Table 1, for Eq. (5).

elasticity [22]. The discretization utilized 216 nodes. The new feature here is that computation times for a post-processing evaluation, using a Galerkin method [9], for the boundary stress tensor are also provided. The stress calculation also requires an integration over the boundary, including non-singular integral evaluation. As with the Laplace equation, the Green's function for orthotropic elasticity involves logarithms, and thus it is not surprising that the reduction for the nonsingular integrations is quite respectable, 35%. Taking the entire solution into account, however, this number is less spectacular, indicating that the singular integrations are dominating the computation. Note that an adjacent singular integration calculation will always produce some contributions to the matrix diagonal. Thus, these integrations would always be computed in the 'bypassing integrations' scheme, whereas the approach adopted herein, combining $\{E_P,\ E_Q\}$ and $\{E_Q,\ E_P\}$ integrations, should still prove useful. For the orthotropic calculations, a substantial increase in performance clearly requires that the modified method be applied to the adjacent singular integrations.

Routine	Original Time (s)	Modified Time (s)	Percent Reduction
nonsingular	37.36	24.33	34.9
total	139.60	126.82	9.2
with stress	175.72	156.06	11.2

Table 3: A comparison of timing results for the original and modified Symmetric-Galerkin nonsingular integral routines for 2-D orthotropic elasticity, mixed boundary value problem.

3.2 Three dimensions

The modified algorithm for three dimensional problems will be examined using the Laplace equation and a frequency domain formulation for elastic wave scattering [23]. In both cases, a non-symmetric formulation is tested; that is, only the singular equation Eq. (4) is employed in solving the problem.

The timing results for the Laplace equation are presented in Table 4. The test problem was discretized using 1592 nodes and 2196 elements. The new aspect here is that the standard Green's function $G(P,Q) = 1/(4\pi r)$ has been replaced by the modified fundamental solution,

$$G_0(P,Q) = \frac{1}{4\pi}\left[\frac{1}{\|Q-P\|} + \frac{1}{\|Q-P_0\|}\right] . \tag{9}$$

If $P = (x_p, y_p, z_p)$, then P_0 is defined as $P = (x_p, y_p, -z_p)$. This Green's function has zero flux on the $z = 0$ surface, and is therefore useful for problems in which this is the prescribed boundary condition [3]. As indicated by Eq. (9), the nonsingular integration now has two components, and Table 4 presents separate timings for each.

Routine	Original Time (s)	Modified Time (s)	Percent Reduction
nonsingular	1454.7	1037.8	28.7
reflected	1475.7	1085.7	26.4
total	4900.4	4137.8	15.6

Table 4: A comparison of timing results for the original and modified nonsingular integral routines for the 3D Laplace equation. A Green's function incorporating a reflection around $z = 0$ has been employed.

The reductions seen for the non-singular integrations are once again quite respectable, and the result for the total program again indicates that speeding up the adjacent singular integrations should be investigated. As indicated by Table 5, the total computation time result for the elastic wave scattering problem is consistent with Laplace. For these three dimensional problems, the hypersingular kernel is considerably more complicated than the original Green's function, and thus it is expected that implementing the fast procedure will be even more beneficial for the derivative equation.

Routine	Original Time (s)	Modified Time (s)	Percent Reduction
total	1395.1	1150.6	17.5

Table 5: A comparison of timing results for the original and modified nonsingular integral routines for a 3D Elastodynamics problem.

4 Conclusion

The changes to the usual Galerkin implementation that were described herein are quite simple. Nevertheless, these modifications have been shown to have a significant impact on performance. The numerical tests indicate that matrix construction time is reduced somewhere between 20% to 35%, for just the non-singular integrations. Combined with the advantages that symmetry provides in the solution of the linear system, these techniques provide an efficient Galerkin boundary integral algorithm, especially for moderate to large scale problems.

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