ON SOME BENCHMARK RESULTS FOR THE INTERACTION OF A CRACK WITH A CIRCULAR INCLUSION

(Revised version submitted to *ASME Journal of Applied Mechanics* – December, 2006)

A.-V. Phan[1], **L. J. Gray**[2], and **T. Kaplan**[2]

[1] Department of Mechanical Engineering
University of South Alabama
Mobile, AL 36688, U.S.A.

[2] Computer Science and Mathematics Division
Oak Ridge National Laboratory
Oak Ridge, TN 37831, USA

Abstract

Stress intensity factor calculations for crack-inclusion interaction problems are presented. The problems considered include the benchmark problems first discussed by Helsing and Jonsson, and subsequently by Wang, Mogilevskaya and Crouch. The numerical results are obtained using the symmetric-Galerkin boundary element method in conjunction with an improved quarter-point element for evaluating the stress intensity factors by means of the displacement correlation technique. The converged results confirm the accuracy of the previous simulations, and demonstrate that accurate solutions for these interaction problems can be obtained with numerical methods that are applicable in three dimensions.

1. Introduction

The interaction between an arbitrary crack and a circular inclusion is a subject of important interest, and thus has attracted a great deal of contributions from various research groups. However, some numerical benchmark results on the subject have recently been questioned by Helsing and Jonsson (HJ); they showed in Ref. [1] that their converged results for the stress intensity factors (SIFs) K_{I} and K_{II} differ from values published in the literature. Following their challenge to the computational mechanics community to confirm or disprove their findings, Wang, Mogilevskaya and Crouch (WMC) [2] have produced results obtained using a Galerkin boundary integral (GBI) method and a complex variables boundary element method (CVBEM). The WMC solutions agree with the HJ results, even though in one case it is not clear precisely how well: for the problem of a circular arc crack interacting with a circular inclusion (Fig. 1(b)), the HJ results were only shown graphically.

Although the WMC work has substantially confirmed the HJ results, we believe some further discussion is warranted. In our opinion it is important to verify, for eventual applications, that these problems can also be accurately solved using 'standard' numerical techniques, *i.e.* techiques that are directly applicable in three dimensions. Note that the integral equation elasticity formulations in HJ and WMC rely upon complex variable methods, and are therefore decidedly two-dimensional algorithms. Moreover, some of the numerical approximations employed in their analyses, *e.g.* global function approximations rather than local element interpolations, are not routinely used in three

dimensions. This is *not* to say that these are not good methods – quite the contrary, we think these approaches are remarkably accurate – just not directly extendable to three dimensions. Thus, in addition to confirming the HJ and WMC solutions, the purpose of this paper is to establish that a general algorithm can successfully solve these types of problems.

For the problem of a straight crack interacting with a circular inclusion (Fig. 1(a)), the WMC calculations did not agree with (a likely misprint in) the HJ result, and in this note we confirm the WMC answer. In addition, by tabulating the results for the circular crack problem shown in Fig. 1(b) we can confirm the WMC calculations.

2. Problem Descriptions and Method of Solution

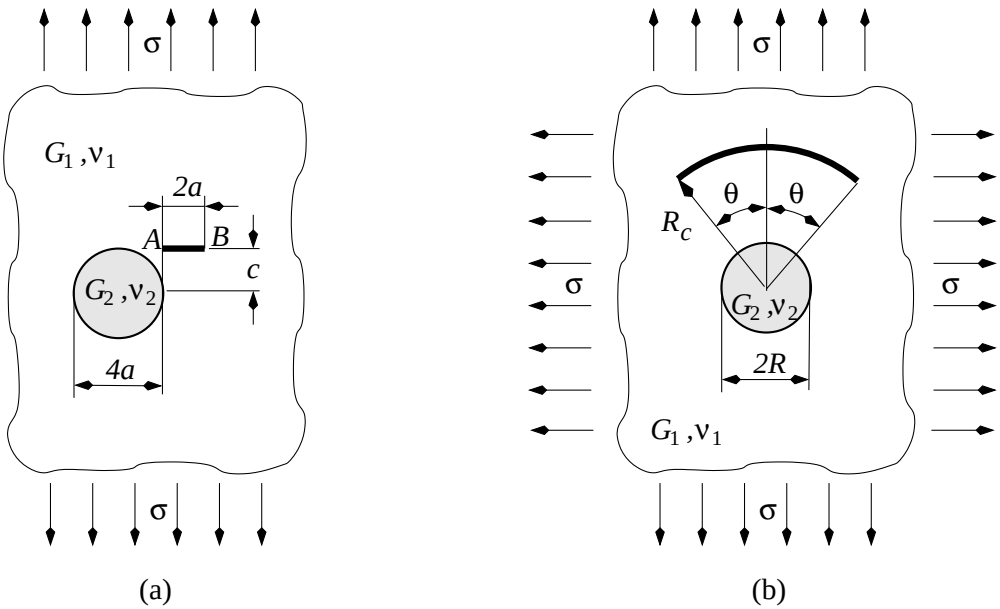

Figure 1: Crack-circular inclusion interaction under remote stress: (a) Straight crack; (b) Circular arc crack.

We consider the two problems discussed in Ref. [1], shown in Figs. 1(a) and 1(b). The straight crack problem (a) was initially studied by Erdogan, Gupta and Ratwani [3], while problem (b) involves a circular arc crack and was studied by Cheeseman and Santare [4]. Plane strain is assumed. For both problems, the shear moduli of the matrix and the inclusion are respectively $G_1 = 1$ and $G_2 = 23$, and the corresponding Poisson's ratios are $\nu_1 = 0.35$ and $\nu_2 = 0.3$.

The calculations presented herein utilize a (more or less standard) quadratic element symmetric-Galerkin boundary integral analysis [5]. However, the stress intensity factor computation is based upon the modified quarter point (MQP) element [6]. The basic idea of the MQP is to modify the quadratic shape functions at the crack tip (adding an appropriate cubic term) in order that the crack opening displacement satisfy a known constraint: the term that is linear in distance to the tip must vanish [7]. This element has been shown to yield highly accurate SIF values (even by means of the simple displacement correlation technique) for standard crack problems, and this work establishes that this carries over to the crack/inclusion problems considered herein. The MQP approach of modifying the crack tip shape functions extends directly to three dimensions; alternatively, the constraint on the tip displacement can be incorporated directly vy employing

an appropriate expansion, as in [8]. The highly accurate results obtained by these authors would indicate that the MQP in 3D will also be very successful.

3. Results and Discussions

For both problems, the straight/arc crack and the matrix-inclusion interface are discretized into uniform elements. All of the results reported in this section are convergent with respect to mesh refinement. For example, Fig. 2 shows the convergence of the normalized SIFs F_{I} and F_{II}, defined as $F_{\mathrm{I}} = K_{\mathrm{I}}/(\sigma\sqrt{\pi a})$ and $F_{\mathrm{II}} = K_{\mathrm{II}}/(\sigma\sqrt{\pi a})$, for problem (a) with $c/a = 1.0$. Here, the matrix/inclusion interface is meshed using $n_i = 68$ elements. It can be seen that the solution for the SIFs at both crack tips A and B converges quickly as the number of crack elements n_c approaches 10.

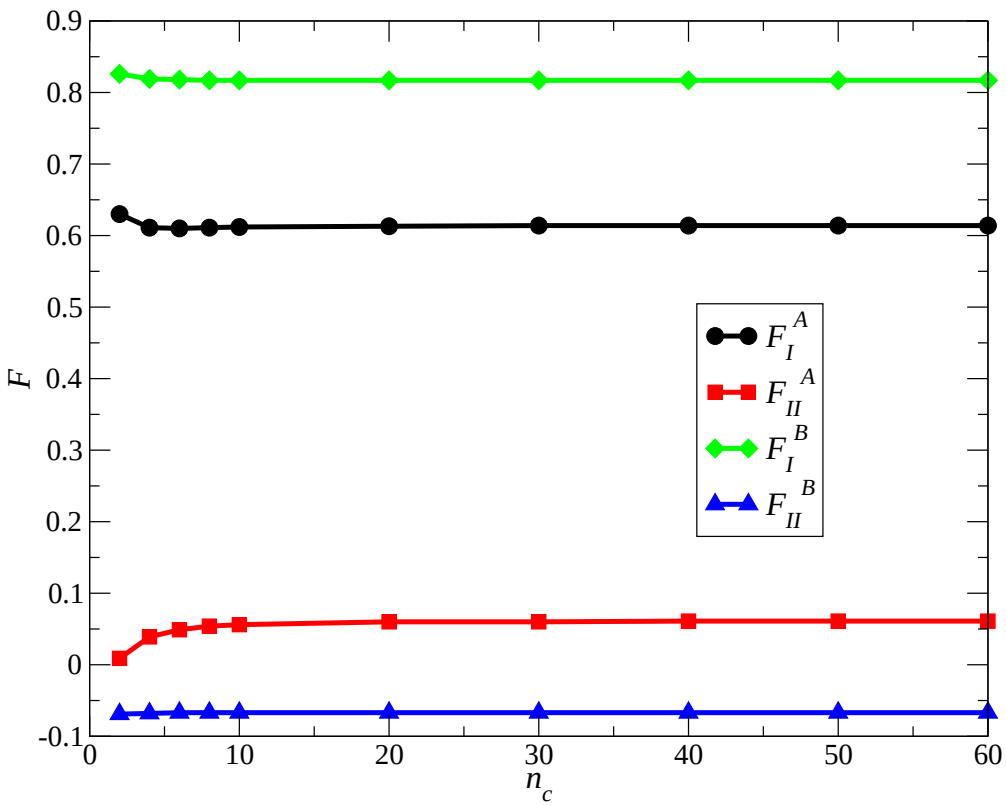

Figure 2: Convergence of F for $c/a = 1.0$

3.1. A straight crack interacting with a circular inclusion

Our symmetric-Galerkin boundary integral results using standard (SQP) and modified quarter-point elements for the normalized SIFs at crack tips A and B are presented in Tables 1 and 2, respectively, together with the WMC, HJ and EGR results. Here, it is important to observe that the additional accuracy provided by the MQP is essential in matching the results in HJ and WMC. Finally, the significant differences in the numerical methods provide additional confirmation of the correctness of the SIF results.

As expected, only coarse meshes are needed to obtain the converged solution when a crack tip is not very close to the inclusion boundary. For example, in case $c/a = 8$, $n_c = 8$ and $n_i = 28$ are

required to reach the convergence. Overall, our MQP solutions agree very well with HJ and WMC, and confirm WMC's belief that there is a misprint in the F_{II} HJ result for the case $c/a = 3$: our result (-0.003) is much closer to the WMC value (-0.004) than the HJ (-0.035).

	F_I^A					F_{II}^A				
c/a	SQP	MQP	WMC	HJ	EGR	SQP	MQP	WMC	HJ	EGR
0.3	0.202	0.234	0.236	0.235	0.225	0.089	0.073	0.074	0.073	0.072
0.5	0.338	0.348	0.347	0.347	0.341	0.113	0.102	0.102	0.102	0.101
1.0	0.616	0.614	0.613	0.613	0.613	0.068	0.061	0.061	0.061	0.057
1.5	0.761	0.756	0.755	0.755	0.763	0.015	0.012	0.012	0.012	-0.007
2.0	0.836	0.830	0.830	0.830	0.845	0.020	0.018	0.018	0.018	-0.021
3.0	0.944	0.937	0.936	0.936	0.953	0.068	0.067	0.067	0.067	-0.001
4.0	1.010	1.003	1.003	1.003	1.014	0.080	0.079	0.079	0.079	0.002
8.0	1.049	1.043	1.043	1.043	1.043	0.032	0.032	0.032	0.032	-0.026

Table 1: Normalized stress intensity factors at crack tip A ($F = K/(\sigma\sqrt{\pi a})$)

	F_I^B					F_{II}^B				
c/a	SQP	MQP	WMC	HJ	EGR	SQP	MQP	WMC	HJ	EGR
0.3	0.803	0.790	0.790	0.790	0.784	-0.022	-0.022	-0.023	-0.023	-0.004
0.5	0.803	0.797	0.797	0.797	0.792	-0.037	-0.037	-0.037	-0.037	-0.006
1.0	0.823	0.817	0.817	0.817	0.817	-0.067	-0.067	-0.067	-0.067	-0.005
1.5	0.839	0.833	0.833	0.833	0.839	-0.074	-0.074	-0.074	-0.074	0.008
2.0	0.855	0.850	0.850	0.850	0.860	-0.058	-0.057	-0.058	-0.058	0.034
3.0	0.903	0.898	0.897	0.897	0.905	-0.004	-0.003	-0.004	-0.035	0.089
4.0	0.953	0.948	0.947	0.947	0.951	0.032	0.032	0.032	0.032	0.117
8.0	1.028	1.022	1.022	1.022	1.020	0.032	0.032	0.032	0.032	0.088

Table 2: Normalized stress intensity factors at crack tip B ($F = K/(\sigma\sqrt{\pi a})$)

To further demonstrate the accuracy and robustness of the numerical techniques, the above calculation is modified so that the inclusion and matrix are of the same material. In this case there is a known analytical result (reference??), and the converged MQP numbers in Table 3 are virtually identical to this solution. A minor numerical error for F_I^A in case $c/a = 0.3$ is expected as this

crack tip is very close to the inclusion boundary.

c/a	0.3	0.5	1.0	1.5	2.0	3.0	4.0	8.0
F_I^A	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
F_I^B	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
F_{II}^A	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
F_{II}^B	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 3: Normalized stress intensity factors for the case where the inclusion is of the same material as the matrix (Analytical solution: $F_I = K_I/(\sigma\sqrt{\pi a}) = 1$ and $F_{II} = K_{II}/(\sigma\sqrt{\pi a}) = 0$)

3.2. A circular arc crack interacting with a circular inclusion

For this problem, the SIFs are respectively normalized by the SIFs $K_{\mathrm{I}o}$ and $K_{\mathrm{II}o}$ in the absence of the inclusion which are given by [9]

$$\left\{ \begin{array}{c} K_{\mathrm{I}o} \\ \\ K_{\mathrm{II}o} \end{array} \right\} = \frac{\sigma\sqrt{\pi R_c \sin\theta}}{1 + \left(\sin\frac{\theta}{2}\right)^2} \left\{ \begin{array}{c} \cos\dfrac{\theta}{2} \\ \\ \sin\dfrac{\theta}{2} \end{array} \right\} . \tag{1}$$

Our converged results using the MQP element are listed along with the WMC's solution in Table 4 where a very good agreement can be seen. As mentioned earlier, it is important to numerically confirm the WMC results for this problem as the corresponding HJ results are only available graphically.

4. Conclusions

It has been established that a standard Galerkin boundary integral algorithm, together with a modified quarter point crack tip element, is capable of accurately solving the benchmark problems discussed by Helsing and Jonsson, and also by Wang *et al.*. The accuracy is confirmed not only through agreement with these previous results, but also by observed convergence with mesh refinement. This algorithm should therefore be equally successful for three dimensional applications.

Acknowledgements

This research was supported in part by the U.S. Army Research Office Grant #W911 NF-04-1-0257, and by the Applied Mathematical Sciences Research Program of the Office of Mathematical, Information, and Computational Sciences, the U.S. Department of Energy under contract DE-AC05-00OR22725 with UT-Battelle, LLC.

$\frac{R_c}{R}$	$\theta = 30°$				$\theta = 45°$				$\theta = 75°$			
	F_I		F_{II}		F_I		F_{II}		F_I		F_{II}	
	MQP	WMC	MQP	WMC	MQP	WMC	MQP	WMC	MQP	WMC	MQP	WMC
1.1	0.919	0.919	1.494	1.494	0.928	0.928	1.393	1.393	0.955	0.955	1.363	1.363
1.2	0.944	0.944	1.353	1.353	0.962	0.962	1.284	1.285	1.009	1.009	1.281	1.281
1.5	0.961	0.961	1.202	1.202	0.990	0.990	1.159	1.159	1.056	1.056	1.160	1.160
2.0	0.972	0.972	1.103	1.104	0.992	0.992	1.092	1.092	1.059	1.059	1.083	1.084
3.0	0.986	0.986	1.039	1.040	0.992	0.992	1.042	1.043	1.039	1.039	1.036	1.037
4.0	0.992	0.992	1.120	1.021	0.995	0.994	1.024	1.024	1.025	1.025	1.020	1.021
5.0	0.995	0.995	1.012	1.013	0.996	0.996	1.015	1.016	1.017	1.017	1.013	1.014
6.0	0.996	0.997	1.008	1.009	0.997	0.997	1.010	1.011	1.012	1.012	1.009	1.010

Table 4: Normalized stress intensity factors at the crack tips ($F = K/K_o$)

References

[1] Helsing, J., and Jonsson, A., 2002, "On the accuracy of benchmark tables and graphical results in the applied mechanics literature," ASME Journal of Applied Mechanics, **69**, pp. 88–90.

[2] Wang, J., Mogilevskaya, S.G., and Crouch, S.L., 2003, "Benchmark results for the problem of interaction between a crack and a circular inclusion," ASME Journal of Applied Mechanics, **70**, pp. 619–621.

[3] Erdogan, F., Gupta, G.D., and Ratwani, M., 1974, "Interaction between a circular inclusion and an arbitrarily oriented crack," ASME Journal of Applied Mechanics, **41**, pp. 1007–1013.

[4] Cheeseman, B.A., and Santare, M.H., 2000, "The interaction of a curved crack with a circular inclusion," International Journal of Fracture, **103**, pp. 259–277.

[5] Bonnet, M., 1995, *Boundary Integral Equation Methods for Solids and Fluids*, John Wiley & Sons, England.

[6] Gray, L.J., Phan, A.-V., Paulino, G.H., and Kaplan, T., 2003, "Improved quarter-point crack tip element," Engineering Fracture Mechanics, **70**, pp. 269–283.

[7] Gray, L.J., and Paulino, G.H., 1998, "Crack tip interpolation, revisited," SIAM J. Applied Mathematics, **58**, pp. 428–455.

[8] Li, S., Mear, M.E., and Xiao, L., 1998, "Symmetric weak form integral equation method for three-dimensional fracture analysis," Comp. Meth. Appl. Mech. Engng., **151**, pp. 435–459.

[9] Tada, H., Paris, P., and Irwin, G., 1973, *The Stress Analysis of Cracks Handbook*, Del Research Corporation, Hellertown, Pennsylvania.