On Adaptive Mesh Generation in Two-dimensions

A quadratic model assumes locally a quadratic Taylor expansion. Consider interpolation error over piecewise-linear triangles. Error curves form a family of conics. Ellipses if convex surface and hyperbolas if saddle shaped case. Bilinear if degenerate or linear case. Assume locally $f$ is well approximated by quadratic Taylor

$$\nabla \left[ \begin{array}{cc} \partial_x^2 & 0 \\ 0 & \partial_y^2 \end{array} \right] = S$$

where $\partial$ is the Laplacian operator. Transformation consists of rotation to align axes through eigenvectors, followed by a rescaling based on eigenvalues.

$$\nabla \left[ \begin{array}{cc} \frac{1}{\lambda} \partial_x & 0 \\ 0 & \frac{1}{\lambda} \partial_y \end{array} \right] \nabla = H$$

Longest edge over isotropic space.

Maximm error is attained either at one of the midpoints of $\partial = (\partial \partial^2 - x^2)^{-1} - \frac{1}{2} \frac{\partial^2 + \partial^2 x + \partial^2 y}{x^2 + y^2}$

Enter expression over isotropic space $\partial x^2$ simplifies to $\partial x^2$ as $\partial x^2$ holds curvature information and can be interpreted as a local metric tensor for approximation errors.

Eigenvalue decomposition of Hessian matrix $\partial^2 x^2$ based on $\partial x^2 S = \partial x^2 \partial x^2$
Exercises only minimal capability, with exact solution supplied.

Continuous transformation

PLTMG


Classical result in differential geometry lead to a sufficient condition

\[ 0 = \varepsilon \psi \lambda \eta + \varepsilon \eta \lambda \psi \]

for some constants \( \lambda, \psi, \eta \).

Continuous transformation

Mesh refinement

PLTMG uses Hessian information to estimate error at mid point

\[ \text{error of vertex is maximum error associated with neighbor elements.} \]

Mesh unrefinement

Valid information

From heap: Kemeny recursion applied to neighborhoods to form a

longest edge bisection of Rivara. Elements with highest error selected.

\[ \begin{bmatrix} f_{\bar{h}} - f_{h} \\ f_{x} - f_{x} \end{bmatrix} \begin{bmatrix} h_{n} \\ h_{x} \end{bmatrix} \begin{bmatrix} f_{\bar{h}} - f_{h} \\ f_{x} - f_{x} \end{bmatrix} \]

resulting vacant region is retriangulated.

Vertex with highest error selected from heap and removed from mesh.

Error of vertex is maximum error associated with neighbor elements.

Mesh unrefinement

PLTMG
Mesh smoothing to reduce the approximation error by adjusting coordinates of vertices.

Gauss-Seidel-like iteration where each vertex is locally adjusted while keeping all neighboring vertices fixed.

Boundary and interior nodes are constrained to move along constant directions.

Example I: A logarithmic singularity at

\[ \frac{z}{(\log r - \bar{a}) + (0x - x)\eta} = (\bar{r} \cdot x) \bar{r} \]

Coordinate transformation is

\[ \frac{z}{(\log r - \bar{a}) + (0x - x)\eta} = (M \cdot x) f \\
\frac{(z \cdot 0 - \bar{a} \cdot x)}{(0 \cdot 0) \cdot x} = (0 \cdot 0) f \]

Example 1: A logarithmic singularity at

**Computer Science and Mathematics Division ORNL**
Final mesh by PLTMG over isotropic space for Example 1.

Table 1: Summary of results for Example 1.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Number of Elements</th>
<th>Minimum</th>
<th>Median</th>
<th>Maximum</th>
<th>Percentile</th>
<th>Error</th>
<th>Median</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Percentile</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh II</td>
<td>923</td>
<td>3.44e-06</td>
<td>3.44e-06</td>
<td>3.44e-06</td>
<td>3.44e-06</td>
<td>923</td>
<td>3.44e-06</td>
<td>3.44e-06</td>
<td>3.44e-06</td>
<td>3.44e-06</td>
<td>923</td>
</tr>
<tr>
<td>PLTMG</td>
<td>1.53e-04</td>
<td>4.40e-04</td>
<td>6.35e-04</td>
<td>1.15e-03</td>
<td>1897</td>
<td>PLTMG</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 1: A near singularity at

\[ \frac{\varepsilon(0\hat{R} - \hat{R}) + \varepsilon(0\hat{R} - x)}{0\hat{R} - \hat{R}} \leq (\hat{R} \cdot x) \hat{R} \]

\[ \left( \frac{\varepsilon(0\hat{R} - \hat{R}) + \varepsilon(0\hat{R} - x)}{0\hat{R} - \hat{R}} - (\hat{R} \cdot x) \right)^{\frac{1}{2}} \leq (\hat{R} \cdot x) \hat{R} \]

Example 2
Figure 4: Regular mesh transformed to isotropic space for Example 2.

Figure 5: Final mesh by PLTMG over isotropic space for Example 2.

Figure 6: Final mesh produced by PLTMG for Example 2.

Table 2: Summary of results for Example 2.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Error (Min)</th>
<th>Percentile Error (Min)</th>
<th>Error (Median)</th>
<th>Percentile Error (Median)</th>
<th>Error (Max)</th>
<th>Percentile Error (Max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh I</td>
<td>4.71e-02</td>
<td>1.30e-02</td>
<td>4.71e-02</td>
<td>1.30e-02</td>
<td>4.71e-02</td>
<td>1.30e-02</td>
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<tr>
<td>Mesh II</td>
<td>2.41e-02</td>
<td>1.30e-02</td>
<td>2.41e-02</td>
<td>1.30e-02</td>
<td>2.41e-02</td>
<td>1.30e-02</td>
</tr>
<tr>
<td>Mesh III</td>
<td>6.93e-04</td>
<td>1.30e-02</td>
<td>6.93e-04</td>
<td>1.30e-02</td>
<td>6.93e-04</td>
<td>1.30e-02</td>
</tr>
</tbody>
</table>
Example 3

A more severe near singularity at

\[
\begin{align*}
\frac{\epsilon \zeta((\mathbf{0} - \mathbf{n}) + \mathbf{0} \mathbf{x} - \mathbf{x})}{\zeta(\mathbf{0} - \mathbf{n}) (\mathbf{0} \mathbf{x} - \mathbf{x})} \sum \mathbf{z} &= \langle \mathbf{r}, \mathbf{x} \rangle \\
\left( \frac{\epsilon \zeta((\mathbf{0} - \mathbf{n}) + \mathbf{0} \mathbf{x} - \mathbf{x})}{\zeta(\mathbf{0} \mathbf{x} - \mathbf{x}) - \zeta(\mathbf{0} - \mathbf{n})} + 1 \right) \sum \mathbf{z} &= \langle \mathbf{r}, \mathbf{x} \rangle \\
\end{align*}
\]

Coordinate transformation is

\[
\begin{align*}
\mathbf{g} - (\zeta(\mathbf{0} - \mathbf{n}) + \zeta(\mathbf{0} \mathbf{x} - \mathbf{x})) &= \mathbf{H} \\
\zeta(\mathbf{0} - \mathbf{n}) + \zeta(\mathbf{0} \mathbf{x} - \mathbf{x}) &= \mathbf{H} \\
\frac{\mathbf{H}}{\sqrt{\epsilon \zeta((\mathbf{0} - \mathbf{n}) + \mathbf{0} \mathbf{x} - \mathbf{x})}} &= \langle \mathbf{r}, \mathbf{x} \rangle \\
\end{align*}
\]

Figure 7: Regular mesh transformed to isotropic space for Example 3.

Figure 8: Final mesh over isotropic space for Example 3.

Figure 9: Final mesh produced by PLTMG over isotropic space for Example 3.
The results suggest the solution adaptive methodology employed in PLTMG can be seen as a more effective approach than that of Mesh II. If we consider the median and 90 percentile errors as the basis for comparison, then PLTMG and Mesh I deviate by at most a factor of 2.

The results suggest the solution adaptive methodology employed in PLTMG can be seen as a more effective approach than that of Mesh II. If we consider the median and 90 percentile errors as the basis for comparison, then PLTMG and Mesh I deviate by at most a factor of 2.

Table 3: Summary of results for Example 3.

<table>
<thead>
<tr>
<th></th>
<th>Mesh I</th>
<th>PLTMG</th>
<th>Mesh II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements</td>
<td>1892</td>
<td>916</td>
<td>916</td>
</tr>
<tr>
<td>Minimum error</td>
<td>1.54e-01</td>
<td>6.69e-03</td>
<td>1.51e+00</td>
</tr>
<tr>
<td>Maximum error</td>
<td>1.54e-01</td>
<td>6.69e-03</td>
<td>1.51e+00</td>
</tr>
<tr>
<td>Median error</td>
<td>1.54e-01</td>
<td>6.69e-03</td>
<td>1.51e+00</td>
</tr>
</tbody>
</table>