On Optimal Bilinear Quadrilateral Meshes

Eduardo D'Azevedo
Computer Science and Mathematics Division
Oak Ridge National Laboratory
http://www.epm.ornl.gov/~efdazedo

Summary

- Two types of asymptotically optimal bilinear quadrilateral meshes for minimizing the maximum interpolation errors.
- For convex data function, the error for each element is approximately constant.
- For saddle-shaped data function, an $O(h^3)$ convergence rate may be possible.
- Both meshes are generated from a uniform square mesh in the 'isotropic' space.

Overview

- We motivate the discussion with the equidistribution criteria for an optimal mesh in one-dimension.
- We use a simple Quadratic model to derive an error model for bilinear interpolation.
- We show that a square over the 'isotropic' space is the optimal shape.
- We mention a classical result in differential geometry to generate the coordinate transformation.
- Numerical results are presented that illustrate equidistribution of error and $O(h^3)$ superconvergence.

Optimal mesh in one dimension

- Optimal placement of vertices for piecewise linear interpolation by equidistribution of density function $\Phi(x)$

$$\int_{x_i}^{x_{i+1}} \Phi(x) \approx \text{constant}, \quad \Phi(x) = \sqrt{|f''(x)|}\,.$$

- Let the interpolation error for a data function $f(x)$ over $[a,b]$ $(E(a) = E(b) = 0)$ be

$$\begin{aligned} E(x) &= (x-a)(x-b) f_2/2\,, \\ E(x_c + dx) &= E(x_c) + (dx)^2 f_2/2\,, \quad x_c = (a+b)/2 \\ &= E(x_c) + \epsilon d\tilde{x}^2/2\,, \end{aligned}$$

where $\boxed{\epsilon d\tilde{x}^2 = f_2 dx^2}$, $\epsilon = \text{sign}(f_2)$, and $f_2 \approx f''(x_c)$.

- Maximum error is attained at x_c and is related to $d\tilde{x}^2/2$.
- A uniform mesh in $\tilde{x}$-space leads to an optimal mesh in the original x-space.
- Integrating $\boxed{d\tilde{x} = \sqrt{|f''(x)|}dx}$ yields

$$\tilde{x}(x) = \int_{x_0}^{x} \sqrt{|f''(t)|}dt + \tilde{x}(x_0)\ .$$

- A uniform mesh in $\tilde{x}$-space yields the equidistributing property

$$\begin{aligned} \text{constant} &= d\tilde{x}_i = \tilde{x}_{i+1} - \tilde{x}_i = \tilde{x}(x_{i+1}) - \tilde{x}(x_i)\ , \\ &= \int_{x_0}^{x_{i+1}} \sqrt{|f''(t)|}dt - \int_{x_0}^{x_i} \sqrt{|f''(t)|}dt\ , \\ &= \int_{x_i}^{x_{i+1}} \sqrt{|f''(t)|}dt\ . \end{aligned}$$

Simple quadratic model

- Assume data function $f(x,y)$ is well approximated by

$$\begin{aligned} f(x,y) &= f(x_c + dx, y_c + dy) \\ &\approx f(x_c,y_c) + \nabla f(x_c,y_c)[dx,dy] + \tfrac{1}{2}[dx,dy]H[dx,dy]^t \end{aligned}$$

where matrix H is the Hessian.

- The function is ‘convex’ if $\det(H) > 0$ and ‘saddle-shaped’ if $\det(H) < 0$.

Isotropic transformation

- Let H be diagonalizable as

$$\begin{aligned} H &= Q^t \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} Q = S^t \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix} S, \\ S &= \begin{bmatrix} \sqrt{|\lambda_1|} & 0 \\ 0 & \sqrt{|\lambda_2|} \end{bmatrix} Q, \end{aligned}$$

where $\epsilon = \text{sign}(\det(H))$, and Q is orthogonal, $Q^tQ = I$.

- Transformation S is a rotation to align eigenvectors along the coordinates axes then followed by a simple scaling.
- Let $[\tilde{x},\tilde{y}]^t = S[x,y]^t$, under this transformation S,

$$[dx,dy]H[dx,dy]^t = (d\tilde{x})^2 + \epsilon(d\tilde{y})^2.$$

- Over the transformed space $(\tilde{x}(x,y), \tilde{y}(x,y))$, the Hessian matrix is reduced to a simple form with no preference for any direction. We shall call $(\tilde{x},\tilde{y})$ the ‘isotropic’ space.

Error for Bilinear Quadrilateral

- Basis defined over $(p,q) \in [0,1] \times [0,1]$

$$x(p,q) = \sum_{i=1}^{i=4} x_i \phi_i(p,q), \quad y(p,q) = \sum_{i=1}^{i=4} y_i \phi_i(p,q)$$
$$E(p,q) = \sum_{i=1}^{i=4} f(x_i,y_i)\phi_i(p,q) - f(x(p,q),y(p,q))$$

- Error expression for a *general* quadrilateral is quite complicated.

- The interpolation error for a convex *parallelogram* with a quadratic data function has a simple form with $(p_c,q_c) = (\frac{1}{2},\frac{1}{2})$,

$$E(p,q) = E_Q - \frac{1}{2}(\mu_1(p-p_c)^2 + \mu_2(q-q_c)^2)$$

where

$$[u_x,u_y] = [x_2-x_1, y_2-y_1], \quad [v_x,v_y] = [x_4-x_1, y_4-y_1]$$
$$\mu_1 = [u_x,u_y]H[u_x,u_y]^t, \quad \mu_2 = [v_x,v_y]H[v_x,v_y]^t$$
$$E_Q = E(p_c,q_c) = \frac{1}{8}(\mu_1+\mu_2)$$

Optimal Quadrilateral Shape

- Consider the error attained at the centroid of a *general* convex quadrilateral.

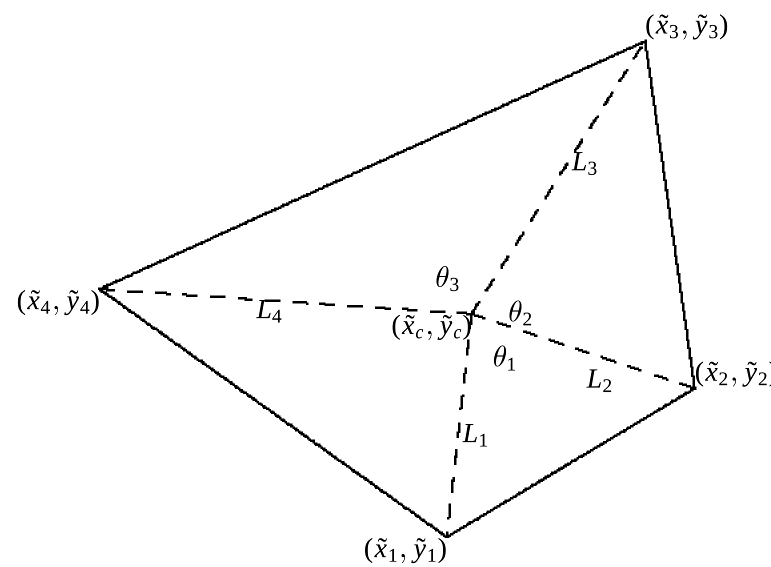
$$E_M = \frac{1}{4}\sum_{i=1}^{i=4} f(x_i,y_i) - f(x_c,y_c)$$

where

$$[x_c,y_c] = [(x_1+x_2+x_3+x_4)/4, (y_1+y_2+y_3+y_4)/4]$$

- The expression can be simplified over the isotropic space,

$$E_M = \frac{1}{8}\sum_{i=1}^{i=4}((\tilde{x}_i^2+\tilde{y}_i^2) - (\tilde{x}_c^2+\tilde{y}_c^2)) = \frac{1}{8}(L_1^2+L_2^2+L_3^2+L_4^2)$$



- Area of quadrilateral over isotropic space is

$$\text{Area} = \frac{1}{2}(L_1 L_2 \sin(\theta_1) + L_2 L_3 \sin(\theta_2) + L_3 L_4 \sin(\theta_3)) - L_4 L_1 \sin(\theta_1 + \theta_2 + \theta_3))$$

- Since transformation S is a rotation and scaling, area over the isotropic space is scaled by $\sqrt{|\lambda_1 \lambda_2|} = \sqrt{|\det(H)|}$ (intrinsic to H).
- By calculus, we can show the ratio E_M/Area is minimized and *attained* by a square

$$L_1 = L_2 = L_3 = L_4, \quad \theta_1 = \theta_2 = \theta_3 = \pi/4, \quad E_M = \frac{L^2}{2}, \quad \text{Area} = L^2$$

- Hence the most efficient shape for all *general* convex bilinear quadrilaterals is a square over the isotropic space.

Error for Square

- The error expression for a parallelogram is

$$E(p,q) = \frac{1}{8}(\mu_1 + \mu_2) - \frac{1}{2}(\mu_1 (p - p_c)^2 + \mu_2 (q - q_c)^2).$$

- For a square in isotropic space,

$$\begin{aligned}
[\tilde{u}_x, \tilde{u}_y] &= [L, 0], \quad [\tilde{v}_x, \tilde{v}_y] = [0, L], \\
\mu_1 &= \tilde{u}_x^2 + \epsilon \tilde{u}_y^2 = L^2, \quad \mu_2 = \tilde{v}_x^2 + \epsilon \tilde{v}_y^2 = \epsilon L^2, \\
E(p,q) &= \frac{L^2}{8}\left((1+\epsilon) - 4((p-p_c)^2 + \epsilon (q-q_c)^2)\right) \\
&= \frac{L^2}{2}((q - \tfrac{1}{2})^2 - (p - \tfrac{1}{2})^2) \quad \text{if } \epsilon = -1.
\end{aligned}$$

- The maximum error is proportional to L^2 and attained either at the center ($\epsilon = 1$) or at a midpoint of an edge ($\epsilon = -1$).

Exact Fit

- The error expression for a parallelogram is

$$E(p,q) = \frac{1}{8}(\mu_1 + \mu_2) - \frac{1}{2}(\mu_1 (p - p_c)^2 + \mu_2 (q - q_c)^2).$$

- Assume $f(x,y)$ is saddle-shaped ($\det(H) < 0, \epsilon = -1$), then both μ_1 and μ_2 vanish for

$$[\tilde{u}_x, \tilde{u}_y] = [L, L] \quad \text{and} \quad [\tilde{v}_x, \tilde{v}_y] = [-L, L]$$

which corresponds to a square rotated by $\pi/4$.

- This suggests an 'exact fit' ($E(p,q) \equiv 0$) and the simple quadratic model is inadequate to fully capture the error properties.

Differential Geometry

- The 'isotropic' space $(\tilde{x}(x,y), \tilde{y}(x,y))$ has the property

$$[dx, dy] H [dx, dy]^t = d\tilde{x}^2 + \epsilon d\tilde{y}^2$$

- To generate a *global* coordinate transformation $(\tilde{x}(x,y), \tilde{y}(x,y))$ we interpret H as a metric tensor

$$\begin{aligned}
h_{11} &= \frac{\partial^2}{\partial x^2} f(x,y) = \left(\frac{\partial \tilde{x}}{\partial x}\right)^2 + \epsilon \left(\frac{\partial \tilde{y}}{\partial x}\right)^2, \\
h_{12} &= \frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial \tilde{x}}{\partial x}\frac{\partial \tilde{y}}{\partial y} + \epsilon \frac{\partial \tilde{y}}{\partial x}\frac{\partial \tilde{x}}{\partial y}, \\
h_{22} &= \frac{\partial^2}{\partial y^2} f(x,y) = \left(\frac{\partial \tilde{x}}{\partial y}\right)^2 + \epsilon \left(\frac{\partial \tilde{y}}{\partial y}\right)^2.
\end{aligned}$$

- Classical result in differential geometry for characterizing a ‘flat’ space gives the condition for finding $(\tilde{x}(x,y), \tilde{y}(x,y))$ as

$$0 = \Gamma = \det \begin{pmatrix} h_{11} & \frac{\partial h_{11}}{\partial x} & \frac{\partial h_{11}}{\partial y} \\ h_{12} & \frac{\partial h_{12}}{\partial x} & \frac{\partial h_{12}}{\partial y} \\ h_{22} & \frac{\partial h_{22}}{\partial x} & \frac{\partial h_{22}}{\partial y} \end{pmatrix}.$$

- Sufficient condition is

$$K_1 h_{11} + K_2 h_{12} + K_3 h_{22} = 0$$

for some constants K_1, K_2, K_3.

Numerical Results

- Mesh generated over $[0,1] \times [0,1]$. Generate only elements entirely within domain (ignore distortion at boundary).
- Mesh I corresponds to a uniform mesh of squares in ‘isotropic’ space and equidistributes the error.
- Mesh I shows expect $O(h^2)$ convergence. A 4X increase in elements leads to a 4X decrease in error.
- Mesh II corresponds to a rotated uniform mesh of squares.
- Mesh II is much more accurate than Mesh I and show $O(h^3)$ superconvergence property. A 4X increase in elements leads to a 8X decrease in error.

Example 1

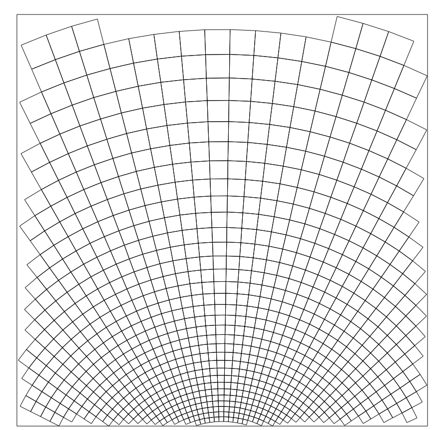
- A logarithmic singularity at $(x_0, y_0) = (0.5, -0.2)$

$$f(x,y) = \ln((x-x_0)^2 + (y-y_0)^2)/2$$

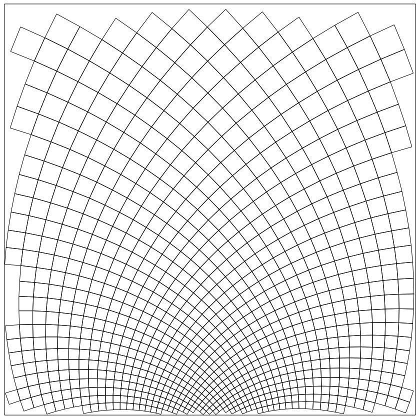
- Note the mesh is radially symmetric.

	Minimum error	Median error	90 percentile	Maximum error	Number of elements
Mesh I	3.56e-04	3.56e-04	3.56e-04	3.56e-04	918
Mesh II	3.44e-06	3.44e-06	3.44e-06	3.44e-06	923

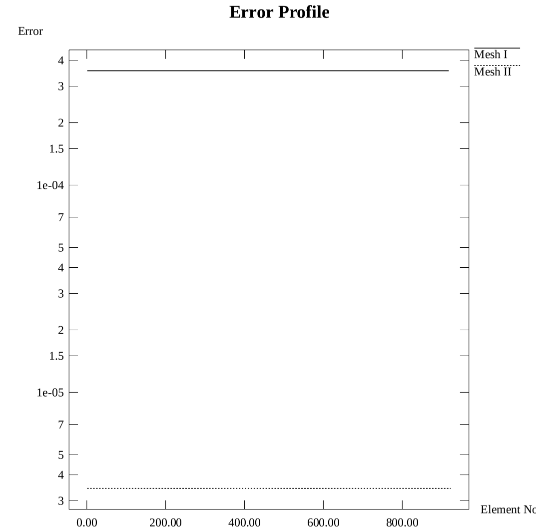
Mesh I for Example 1



Mesh II for Example 1



Results for Example 1



Results for Example 1

	Minimum error	Median error	90 percentile	Maximum error	Number of elements
Mesh I	3.56e-04	3.56e-04	3.56e-04	3.56e-04	918
Mesh I	8.90e-05	8.90e-05	8.90e-05	8.90e-05	3841
Mesh I	2.22e-05	2.22e-05	2.22e-05	2.22e-05	15674
Mesh II	3.44e-06	3.44e-06	3.44e-06	3.44e-06	923
Mesh II	4.30e-07	4.30e-07	4.30e-07	4.30e-07	3847
Mesh II	5.37e-08	5.37e-08	5.37e-08	5.37e-08	15695

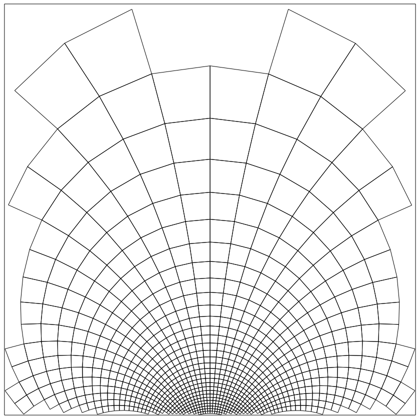
Example 2

- A near singularity at $(x_0, y_0) = (0.5, -0.2)$

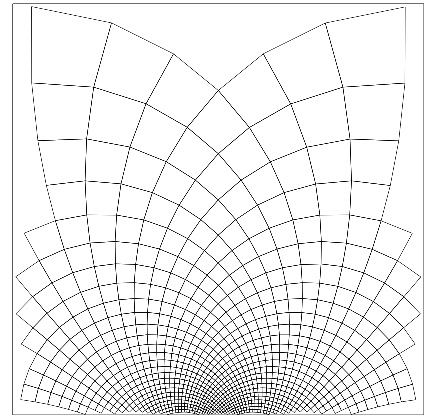
$$f(x,y) = \frac{(x-x_0)^2-(y-y_0)^2}{((x-x_0)^2+(y-y_0)^2)^2}$$

	Minimum error	Median error	90 percentile	Maximum error	Number of elements
Mesh I	1.30e-02	1.30e-02	1.30e-02	1.30e-02	920
Mesh II	1.27e-04	1.79e-04	3.18e-04	6.93e-04	921

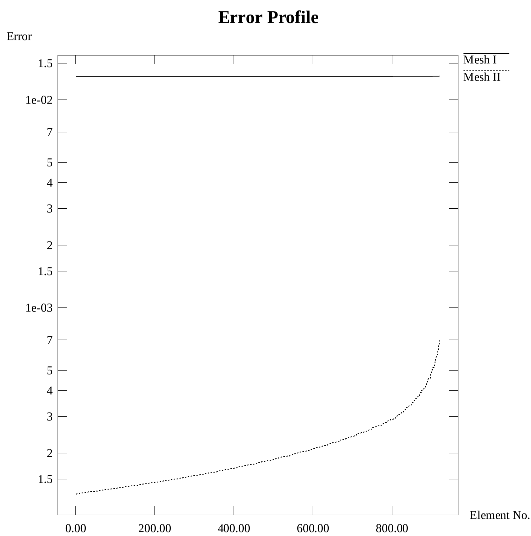
Mesh I for Example 2



Mesh II for Example 2



Results for Example 2



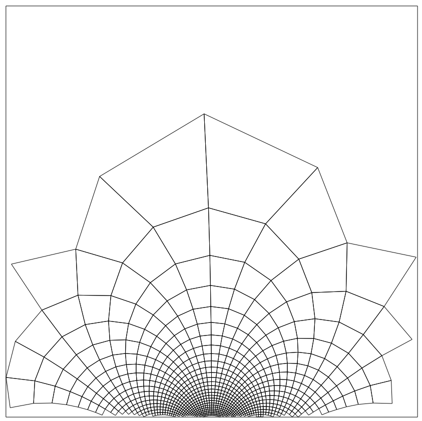
Example 3

- A more severe near singularity at $(x_0, y_0) = (0.5, -0.2)$

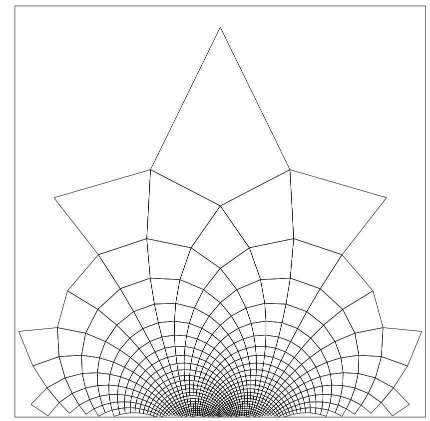
$$f(x,y) = \frac{((x-x_0)^2 + (y-y_0)^2)^2 - 8(x-x_0)^2(y-y_0)^2}{((x-x_0)^2 + (y-y_0)^2)^4}$$

	Minimum error	Median error	90 percentile	Maximum error	Number of elements
Mesh I	4.54e-01	4.54e-01	4.54e-01	4.60e-01	916
Mesh II	3.69e-03	6.69e-03	1.63e-02	9.64e-02	918

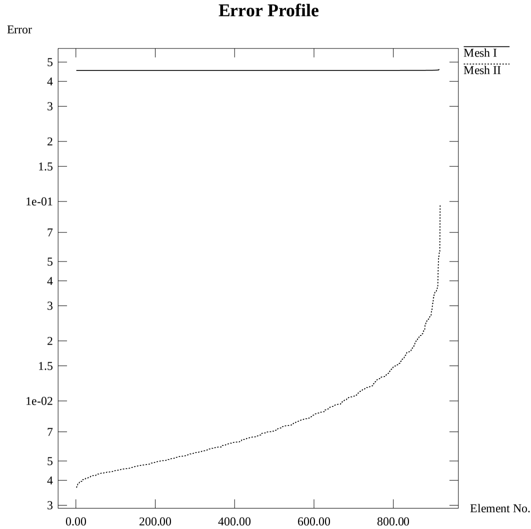
Mesh I for Example 3



Mesh II for Example 3



Results for Example 3



Results for Example 3

	Minimum error	Median error	90 percentile	Maximum error	Number of elements
Mesh I	1.51e+00	1.51e+00	1.52e+00	1.56e+00	255
Mesh I	4.54e-01	4.54e-01	4.54e-01	4.60e-01	916
Mesh I	1.13e-01	1.13e-01	1.14e-01	1.15e-01	3837
Mesh I	2.84e-02	2.84e-02	2.84e-02	2.85e-02	15685
Mesh II	2.36e-02	4.06e-02	9.66e-02	5.09e-01	259
Mesh II	3.69e-03	6.69e-03	1.63e-02	9.64e-02	918
Mesh II	4.52e-04	8.29e-04	2.04e-03	1.44e-02	3834
Mesh II	5.53e-05	1.03e-04	2.54e-04	1.92e-03	15682

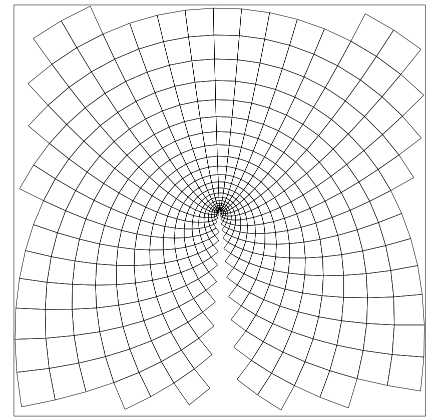
Example 4

- Potential flow around a corner at $(x_0, y_0) = (0.5, 0.5)$, where $n = \pi/\alpha = 16/31$, $\alpha = 2\pi - \pi/16$ is the angle of corner and $\theta = \arctan(y, x)$

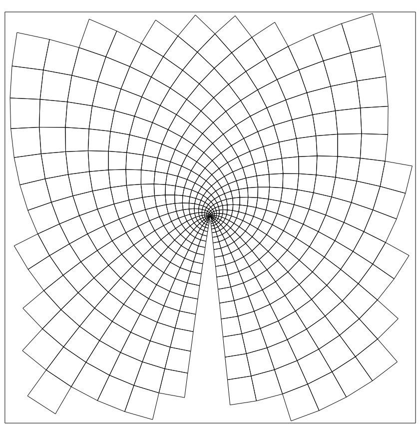
$$f(x,y) = ((x-x_0)^2 + (y-y_0)^2)^{n/2} cos(n\theta)$$

	Minimum error	Median error	90 percentile	Maximum error	Number of elements
Mesh I	4.21e-4	4.21e-4	4.22e-4	4.26e-4	576
Mesh II	5.90e-6	9.90e-6	1.90e-5	3.97e-5	575

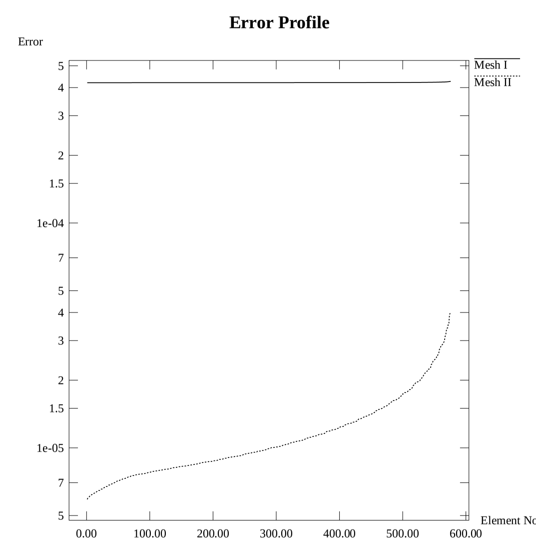
Mesh I for Example 4



Mesh II for Example 4



Results for Example 4



Summary

- Two types of asymptotically optimal bilinear quadrilateral meshes for minimizing the maximum interpolation errors.
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- For saddle-shaped data function, an $O(h^3)$ convergence rate may be possible.
- Both meshes are generated from a uniform square mesh in the ‘isotropic’ space.

Future Directions

- Application in optimal mesh near known singularity, e.g. near crack tip.
- How to generate all quad mesh with prescribed orientation to achieve super-convergence?
- Computing the global coordinate transformation requires high order derivatives. How to extract such information from low order bilinear elements?
- Is there a physical interpretation for $\Gamma = 0$?