On Optimal Bilinear Quadrilateral Meshes

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Summary

Two types of asymptotically optimal bilinear quadrilateral meshes for minimizing the maximum interpolation errors.

For convex data function, the error for each element is approximately constant.

For saddle-shaped data function, an $O(h^3)$ convergence rate may be possible.

Both meshes are generated from a uniform square mesh in the 'isotropic' space.

Numerical results are presented that illustrate equidistribution by equiangular interpolation.

Shape.

We motivate the discussion with the equidistribution criteria for an optimal mesh in one dimension.

We use a simple quadratic model to derive an error model for bilinear interpolation.

We show that a square over the 'isotropic' space is the optimal shape.

We mention a classical result in differential geometry to generate the coordinate transformation.

We prove the discussion with the equidistribution criteria.

Optimal mesh in one dimension

Let the interpolation error for a data function $f(x)$ over $[a,b]$ be

$$
\int_a^b \left[ \frac{E(x)}{f(x)} \right]^2 \, dx
$$

Optimal placement of vertices for piecewise linear interpolation by equidistribution of density function $\Phi(x)$.

Numerical results are presented that illustrate equidistribution by equiangular interpolation.

We mention a classical result in differential geometry to generate the coordinate transformation.

We motivate the discussion with the equidistribution criteria.

Overview

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We show that a square over the 'isotropic' space is the optimal shape.

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Optimal placement of vertices for piecewise linear interpolation by equidistribution of density function $\Phi(x)$.

Numerical results are presented that illustrate equidistribution by equiangular interpolation.

We mention a classical result in differential geometry to generate the coordinate transformation.

We motivate the discussion with the equidistribution criteria.
Maximum error is attained at $x_c$ and is related to $d \tilde{x}^2$.

A uniform mesh in $\tilde{x}$-space leads to an optimal mesh in the $x$-space.

Assume data function $f(x,y)$ is well approximated by

$\begin{align*}
\Delta f &= (\partial^2 f / \partial x^2)(x_c, y_c) \Delta x + (\partial^2 f / \partial y^2)(x_c, y_c) \Delta y \\
&\approx (\partial^2 f / \partial x^2)(x_c, y_c) \Delta x
\end{align*}$

The function is 'convex' if $\det(H) > 0$ and 'saddle-shaped' if $\det(H) < 0$.

Let $\mathbf{H}$ be diagonalizable as $\mathbf{H} = \mathbf{Q} \mathbf{D} \mathbf{Q}^{-1}$, where $\mathbf{D}$ is the diagonal matrix with the eigenvalues $\lambda_1, \lambda_2, \ldots$ and $\mathbf{Q}$ is orthogonal.

Let $\mathbf{S}$ be a rotation to align eigenvectors along the coordinates axes then followed by a simple scaling.

The transformed space $(\tilde{x}(x,y), \tilde{y}(x,y))$ is called the 'isotropic' space.

A uniform mesh in $x$-space yields the equidistributing property $\lambda \approx \lambda / \Delta x$.
Error for Bilinear Quadrilateral of Basis defined over \((p,q)\):

\[
E(p,q) = \sum_{i=1}^{4} \frac{f(x_i,y_i)}{f(x_i,y_i)}
\]

Error expression for a general quadrilateral is quite complicated.

The interpolation error for a convex parallelogram with a quadratic data function has a simple form with \((p_c,q_c)\):

\[
E_{\text{Q}}(p_c,q_c) = \frac{1}{16} \left( (\tilde{x}_1^+ \tilde{y}_1^- + \tilde{x}_1^- \tilde{y}_1^+) \phi \left( \frac{1}{2} \tilde{x}_1 \tilde{y}_1 + \tilde{x}_1 \tilde{y}_1 \right) \right)
\]

where

\[
\phi = \frac{1}{16} \left( \frac{L_1^2}{1} + \frac{L_2^2}{2} + \frac{L_3^2}{3} + \frac{L_4^2}{4} \right)
\]

Optimal Quadrilateral Shape

Consider the error attained at the centroid of a general convex quadrilateral.

The expression can be simplified over the isotropic space:

\[
E_{\text{M}} = \frac{1}{16} \sum_{i=1}^{4} \left( \tilde{x}_i \tilde{y}_i \right)
\]

\[
\tilde{x}_1 \tilde{y}_1 = \left[ x_1 \ x_2 \ x_3 \ x_4 \right] \left[ y_1 \ y_2 \ y_3 \ y_4 \right]^T
\]

\[
\text{Quadilateral}
\]

Error for Bilinear Quadrilateral

\[
\left( \tilde{x}_1 \tilde{y}_1 + \tilde{x}_2 \tilde{y}_2 + \tilde{x}_3 \tilde{y}_3 \right) \frac{1}{16} = \frac{1}{16} \left( \frac{L_1^2}{1} + \frac{L_2^2}{2} + \frac{L_3^2}{3} + \frac{L_4^2}{4} \right)
\]

\[
\text{Quadilateral}
\]

Error for Bilinear Quadrilateral

\[
\left( f \left( \frac{1}{b^2 + d^2} \right) \right) = \frac{1}{16} \left( \frac{L_1^2}{1} + \frac{L_2^2}{2} + \frac{L_3^2}{3} + \frac{L_4^2}{4} \right)
\]

\[
\text{Quadilateral}
\]
Properties.

The error expression for a parallelogram is

\[ \text{Area of quadrilateral over isotropic space is} \]

Hence the most efficient shape for all general convex billinear

\[ \text{Area} \quad \frac{\text{area}}{\text{area}} = \frac{\text{area}}{\text{area}} = \frac{\text{area}}{\text{area}} = \frac{\text{area}}{\text{area}} = \frac{\text{area}}{\text{area}} \]

which corresponds to a square rotated by a square.

\[ \text{By calculus, we can show the ratio area} / \text{area} \]

is minimized and

\[ \text{Since transformation } S \text{ is rotation and scaling area over the} \]

\[ \text{error expression for a parallelogram is} \]

\[ \text{Exact Fit} \]

\[ \text{Differential Geometry} \]

\[ \text{Quadrilateral is a square over the isotropic space.} \]

\[ \text{Hence the most efficient shape for all general convex billinear} \]

\[ \text{Hence the most efficient shape for all general convex bilinear} \]

\[ \text{Exact Fit} \]
Classical result in differential geometry for characterizing a 'flat' space gives the condition for finding \( \tilde{x}(x/y) \) as \( 0 \),

\[
\Gamma \det \begin{pmatrix} 
  h_{11} & h_{12} \\
  h_{12} & h_{22} 
\end{pmatrix}
\]

Sufficient condition is

\( K_1 h_{11} + K_2 h_{12} + K_3 h_{22} = 0 \)

\( \frac{1}{(x, y)} = \frac{1}{(0, 0)} \) a logarithmic singularity at \( (x, y) = (0, 0) \).

\[ f(x,y) = \frac{1}{(x, y)} \]

\( \\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = 0 \)

Mesh I for Example 1

\[ f(\tilde{x}(x/y)) \ln((x, y)\tilde{x}(x/y)) = 2 \]

Mesh I shows superconvergence property. A \( 4x \) decrease in error leads to Mesh II much more accurate than Mesh I and shows \( O(h^4) \) convergence.

Mesh II corresponds to a rotated uniform mesh of squares. Mesh II is much more accurate than Mesh I and shows \( O(h^2) \) convergence. A \( 4x \) decrease in error leads to a \( 4x \) decrease in error.

Mesh I is radially symmetric.

\begin{table}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Mesh & \# of elements & Error & Percentile & Maximum & Median & Minimum \\
\hline
Mesh II & 923 & 3.44e-06 & 3.44e-06 & 3.44e-06 & 3.44e-06 & 3.44e-06 \\
\hline
\end{tabular}
\end{table}

\textbf{Example 1}

\textbf{Numerical Results}

Mesh I corresponds to a uniform mesh of squares in 'isotropic' space and equidistributes the error.

Mesh II is much more accurate than Mesh I and shows superconvergence property. A \( 4x \) decrease in error leads to a \( 4x \) decrease in error.

\( \frac{1}{(x, y)} = \frac{1}{(0, 0)} \) a logarithmic singularity at \( (x, y) = (0, 0) \).

\[ f(\tilde{x}(x/y)) \ln((x, y)\tilde{x}(x/y)) = 2 \]

Note the mesh is radially symmetric.

\[ f(\tilde{x}(x/y)) \ln((x, y)\tilde{x}(x/y)) = 2 \]
Example 2

A near singularity at \((x_0, y_0) = (0.5, -0.2)\).

\[
\frac{\chi(0, y) - \chi(0, y_0)}{\chi(0, y) - \chi(0, y_0)} = (h, x)f
\]

\[
(z, y) = (0, y_0) = (0.5, -0.2)
\]
Example 3

A more severe near singularity at $(x_0, y_0)$

\[
\frac{\varepsilon(\varepsilon(0) - h) + \varepsilon(0) - x)}{\varepsilon(0) - h, (0) - x} = (h, x)
\]

\[
\frac{\varepsilon(0) - h, (0) - x) = (0, 0)^{0.3} - 0.2}
\]

Results for Example 2

<table>
<thead>
<tr>
<th>Mesh I</th>
<th>Mesh II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>918</td>
<td>916</td>
</tr>
<tr>
<td>916</td>
<td>916</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Elements</th>
<th>Error Percentage</th>
<th>Median</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.54e-01</td>
<td>4.54e-01</td>
<td>4.54e-01</td>
<td>4.60e-01</td>
<td>916</td>
</tr>
<tr>
<td>3.69e-03</td>
<td>6.69e-03</td>
<td>1.63e-02</td>
<td>9.64e-02</td>
<td>916</td>
</tr>
</tbody>
</table>

Mesh I for Example 2

Mesh II for Example 2
### Results for Example 3

#### Error Profile

<table>
<thead>
<tr>
<th>Element</th>
<th>Error Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh I</td>
<td>Error</td>
</tr>
<tr>
<td>3</td>
<td>1.00e-02</td>
</tr>
<tr>
<td>4</td>
<td>1.50e-01</td>
</tr>
<tr>
<td>5</td>
<td>2.00e-01</td>
</tr>
<tr>
<td>6</td>
<td>3.00e-02</td>
</tr>
<tr>
<td>7</td>
<td>4.00e-02</td>
</tr>
<tr>
<td>Mesh II</td>
<td>Error</td>
</tr>
<tr>
<td>3</td>
<td>2.00e-02</td>
</tr>
<tr>
<td>4</td>
<td>3.00e-02</td>
</tr>
<tr>
<td>5</td>
<td>4.00e-02</td>
</tr>
<tr>
<td>6</td>
<td>5.00e-02</td>
</tr>
<tr>
<td>7</td>
<td>6.00e-02</td>
</tr>
</tbody>
</table>

#### Minimum, Median, Maximum Number of Error Elements

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Minimum</th>
<th>Median</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mesh I</td>
<td>255</td>
<td>616</td>
<td>3837</td>
</tr>
<tr>
<td>Mesh II</td>
<td>259</td>
<td>918</td>
<td>3834</td>
</tr>
</tbody>
</table>

#### Diagrams

- **Mesh I for Example 3**
- **Mesh II for Example 3**
Example 4

Potential flow around a corner at \((x_0, y_0)\), where

\[
\theta = \arctan\left(\frac{y}{x}\right)
\]

and \(\phi = \tan^{-1}\left(\frac{16}{3}\right) = \tan^{-1}\left(\frac{16}{3}\right) = \frac{16}{3}\), where

\[
(\theta \mu) \cos \phi \left(\frac{\mu - \tilde{\mu}}{\mu - 1} - \frac{1}{16}\right) = \phi \cdot x
\]

Minimum

Median

90

Maximum

Number of
error elements

<table>
<thead>
<tr>
<th>Mesh I</th>
<th>Mesh II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elements</td>
<td>576</td>
</tr>
<tr>
<td>Error</td>
<td>4.21e-4</td>
</tr>
<tr>
<td>Median</td>
<td>4.22e-4</td>
</tr>
<tr>
<td>Phminum</td>
<td>4.26e-4</td>
</tr>
</tbody>
</table>

Mesh I for Example 4

Mesh II for Example 4

Results for Example 4
Summary

Two types of asymptotically optimal bilinear quadrilateral meshes for minimizing the maximum interpolation errors. For convex data function, the error for each element is approximately constant. For saddle-shaped data function, an $O(h^3)$ convergence rate may be possible. Both meshes are generated from a uniform square mesh in the 'isotropic' space.

Future Directions

Application in optimal mesh near known singularity, e.g. near crack tip. How to generate all quad mesh with prescribed orientation to achieve super-convergence. Computing the global coordinate transformation requires high order derivatives. How to extract such information from low order elements? Is there a physical interpretation for $\Gamma_{0}$?