CANONICAL DICHOTOMIZED ALTERNANT DETERMINANTS AND POISSON MIXTURES

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Abstract

For a Poisson mixture of s components ($2s-1$ parameters) approximate asymptotic variances are set up being based on the maximum likelihood covariance matrix. An associated orthogonal system is used in expressions for expansions of logarithmic derivatives of the basic probability function. Linear reductions of the determinants which appear lead to unexpected canonical forms related to dichotomized alternants. A brief account of the binomial mixture is included.

Some keywords: Alternant determinants, alternant identities, binomial mixture, covariance matrix, factorization, maximum likelihood estimators, orthogonal system.

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1 Introduction

We consider maximum likelihood estimators for the parameters of the Poisson finite mixture,

$$P^{(s)}(x, \underline{\theta}, \underline{\pi}) = \sum_{r=1}^{s} \pi_r \frac{e^{-\theta_r}\theta_r^x}{x!}, \quad (\sum \pi_r = 1, 0 < \pi_r < 1, \theta_r > 0, r = 1, \cdots, s, x = 0, 1, \cdots) \tag{1}$$

and the asymptotic variances of estimators. From Kendall (1946), with some regularity conditions, the probability function being $f(x; \theta_1, \theta_2, \cdots, \theta_s)$, sample size n, the asymptotic covariances of the maximum likelihood estimators $(\hat{\underline{\theta}})$ are given by

$$n \cdot Cov(\hat{\theta}_j, \hat{\theta}_k) = \Delta_{jk}/\Delta \quad (j, k = 1, 2, \cdots, s)$$

where Δ is the Hessian determinant

$$\left| \int_{-\infty}^{\infty} \left(\frac{\partial \ln f}{\partial \theta_j}\right) \left(\frac{\partial \ln f}{\theta_k}\right) f(x, \underline{\theta}) dx \right| \tag{2}$$

and Δ_{jk} is the minor of the element in jth row and kth column. The integral in (2) may be replaced by a Stieltjes form, thus including cases of discrete random variates.

We shall obtain canonical approximants to Δ_{jk} and Δ for the Poisson mixture of Poisson probability functions introducing expressions for $\frac{\partial \ln f}{\partial \theta}$ in terms of orthogonal polynomials. High central moments of a single Poisson random variate are required, and although the cumulants are all the same, this does not avoid some complications when central moments are required; for central moments a simple difference-differential equation is however useful. From Kendall (1943),

$$\mu_{r+1} = r\theta\mu_{r-1} + \theta d\mu_r.$$

A closed form for the orthogonal system related to a mixture of weigh function is apparently not known. This poses a problem, but we can sequentially construct a set of cases. Thus, for example, in terms of determinants

$$q_2(x, \underline{\theta}, \underline{\pi}) = \begin{vmatrix} 1 & X & X^2 \\ \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \div \begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix}, \tag{3}$$

$$q_3(x,\underline{\theta},\underline{\pi}) = -\begin{vmatrix} 1 & X & X^2 & X^3 \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{vmatrix} \div \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix}$$

where μ_s is the sth central moment relating to the mixture in (1) and $X = x - \mu_1'$, μ_1' the mean of the Poisson mixture, x being the associated random variable. The denominators in (3) are "integral squares", and

$$E(q_2^2(x)) = \phi_2 = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix} \div \begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix} = W_2/W_1, \qquad (\phi_1 = W_1 = \mu_2)$$

$$E(q_3^2(x)) = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ \mu_3 & \mu_4 & \mu_5 & \mu_6 \end{vmatrix} \div \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix} = W_3/W_2.$$

It will be clear that ϕ involves central moments of the Poisson mixture up to and including μ_{2s}.

2 Logarithmic derivatives

Taking a simple case consider a mixture of 2 components (3 parameters, 3pP) with maximum likelihood estimators $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\pi}_1$. Then

$$\begin{aligned} \frac{1}{\pi_1}\frac{\partial \ln P}{\partial \theta_1} &= \left(\frac{x-\theta_1}{\theta_1}\right)\frac{e^{-\theta_1}\theta_1^x}{x!}/P(x;\underline{\theta},\underline{\pi}), \\ \frac{1}{\pi_2}\frac{\partial \ln P}{\partial \theta_2} &= \left(\frac{x-\theta_2}{\theta_2}\right)\frac{e^{-\theta_2}\theta_2^x}{x!}/P(x;\underline{\theta},\underline{\pi}), \\ \frac{\partial \ln P}{\partial \pi_1} &= \left(\frac{e^{-\theta_1}\theta_1^x}{x!} - \frac{e^{-\theta_2}\theta_2^x}{x!}\right)/P(x;\underline{\theta},\underline{\pi}). \end{aligned}$$

Then the element (1,1) in the Hessian determinant (2) is

$$\pi_1^2 \sum_{x=0}^{\infty} \left(\frac{x-\theta_1}{\theta_1}\right)^2 \left(\frac{e^{-\theta_1}\theta_1^x}{x!}\right)^2 /P(x;\underline{\theta},\underline{\pi})$$

and for (1,3),

$$\pi_1 \sum_{x=0}^{\infty} \left(\frac{x-\theta_1}{\theta_1}\right)\left(\frac{e^{-\theta_1}\theta_1^x}{x!}\right)\left(\frac{e^{-\theta_1}\theta_1^x}{x!} - \frac{e^{-\theta_2}\theta_2^x}{x!}\right) / P(x;\underline{\theta},\underline{\pi}).$$

The associated orthogonal system is

$$\sum_{x=0}^{\infty} q_r(x) q_s(x) P(x;\underline{\theta},\underline{\pi}) = c_r \delta_{r,s}$$

c_r being a positive real, δ the Kronecker delta function.

We introduce approximation series,

$$\begin{aligned}
\left(\frac{x-\theta_1}{\theta_1}\right)\frac{e^{-\theta_1}\theta_1^x}{x!} &\sim A_0 q_0(x) + A_1 q_1(x) + A_2 q_2(x) + A_3 q_3(x), \\
\left(\frac{x-\theta_2}{\theta_2}\right)\frac{e^{-\theta_2}\theta_2^x}{x!} &\sim B_0 q_0(x) + B_1 q_1(x) + B_2 q_2(x) + B_3 q_3(x), \\
\left(\frac{e^{-\theta_1}\theta_1^x}{x!} - \frac{e^{-\theta_2}\theta_2^x}{x!}\right) &\sim C_0 q_0(x) + C_1 q_1(x) + C_2 q_2(x) + C_3 q_3(x).
\end{aligned}$$

Clearly $A_0 = B_0 = C_0 = 0$, and C_1, C_2, C_3 contain the factor $(\theta_1 - \theta_2)$. These approximants are introduced in the Hessian determinant in (2) yielding

$$\begin{aligned}
\Delta &= \pi_1^2 \pi_2^2 \begin{vmatrix} \sum_1^3 A_r^2 \phi_r & \sum_1^3 A_r B_r \phi_r & \sum_1^3 A_r C_r \phi_r \\ \sum_1^3 B_r A_r \phi_r & \sum_1^3 B_r^2 \phi_r & \sum_1^3 B_r C_r \phi_r \\ \sum_1^3 C_r A_r \phi_r & \sum_1^3 C_r B_r \phi_r & \sum_1^3 C_r^2 \phi_r \end{vmatrix} \\
&= \frac{1}{\phi_1 \phi_2 \phi_3} \begin{vmatrix} A_1\phi_1 & A_2\phi_2 & A_3\phi_3 \\ B_1\phi_1 & B_2\phi_2 & B_3\phi_3 \\ C_1\phi_1 & C_2\phi_2 & C_3\phi_3 \end{vmatrix} \begin{vmatrix} A_1\phi_1 & B_1\phi_1 & C_1\phi_1 \\ A_2\phi_2 & B_2\phi_2 & C_2\phi_2 \\ A_3\phi_3 & B_3\phi_3 & C_3\phi_3 \end{vmatrix} \qquad (4)
\end{aligned}$$

i.e.

$$\Delta \sim \frac{\pi_1^2 \pi_2^2}{\phi_1 \phi_2 \phi_3} \begin{vmatrix} A_1\phi_1 & A_2\phi_2 & A_3\phi_3 \\ B_1\phi_1 & B_2\phi_2 & B_3\phi_3 \\ C_1\phi_1 & C_2\phi_2 & C_3\phi_3 \end{vmatrix}^2 = \frac{\pi_1^2 \pi_2^2}{\phi_1 \phi_2 \phi_3} \Delta_3^{*2}. \qquad (5)$$

More terms could be added to the expansion in (4), but the above reduction would not be possible. Paradoxically the more parameters that are involved suggests sharper approximants because we are using an orthogonal system.

The extension of (5) to additional parameter is obvious, but complications arise since the elements are compound determinants. For example,

$$A_1\phi_1 = -\sum_{x=0}^{\infty}\left(\frac{x-\theta_1}{\theta_1}\right)\frac{e^{-\theta_1}\theta_1^x}{x!}\begin{vmatrix} 1 & X \\ \mu_0 & \mu_1 \end{vmatrix} = 1,$$

$$A_2\phi_2 = \sum_{x=0}^{\infty}\left(\frac{x-\theta_1}{\theta_1}\right)\frac{e^{-\theta_1}\theta_1^x}{x!}\begin{vmatrix} 1 & X & X^2 \\ \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \div W_1$$

$$= \begin{vmatrix} 0 & 1 & 1+2\lambda_1 \\ \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix} \div W_1,$$

$$A_3\phi_3 = -\sum_{x=0}^{\infty}\left(\frac{x-\theta_1}{\theta_1}\right)\frac{e^{-\theta_1}\theta_1^x}{x!}\begin{vmatrix} 1 & X & X^2 & X^3 \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{vmatrix} \div W_2$$

$$= \begin{vmatrix} 0 & 1 & 1+2\lambda_1 & 1+3\theta_1+3\lambda_1+3\lambda_1^2 \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{vmatrix} \div W_2.$$

where $\lambda_1 = \theta_1 - \mu_1'$, $\lambda_2 = \theta_2 - \mu_1'$. Hence there is the approximation

$$\Delta_3 \sim \frac{\pi_1^2\pi_2^2}{\phi_1\phi_2\phi_3}(\lambda_1-\lambda_2)^2 \left|\begin{array}{ccc} 1 & \left|\begin{array}{ccc} 0 & 1 & 1+2\lambda_1 \\ \mu_0 & 0 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \end{array}\right| \div W_1 & d^{(3)}(\theta_1) \\ 1 & \left|\begin{array}{ccc} 0 & 1 & 1+2\lambda_2 \\ \mu_0 & 0 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \end{array}\right| \div W_1 & d^{(3)}(\theta_2) \\ 1 & \left|\begin{array}{ccc} 0 & 1 & 1+\lambda_1+\lambda_2 \\ \mu_0 & 0 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \end{array}\right| \div W_1 & d^{(3)}(\theta_1,\theta_2) \end{array}\right|^2$$

where

$$d^{(3)}(\theta_i) = -\left|\begin{array}{cccc} 0 & 1 & 1+2(\theta_i-\mu_1') & 1+3\theta_i+3\lambda_i+3\lambda_i^2 \\ 1 & 0 & \mu_2 & \mu_3 \\ 0 & \mu_2 & \mu_3 & \mu_4 \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{array}\right| \div W_2, \qquad (i=1,2)$$

and

$$d^{(3)}(\theta_1,\theta_2) = -\left|\begin{array}{cccc} 0 & 1 & 1+\lambda_1+\lambda_2 & 1+3(\theta_1+\theta_2-\mu_1')+\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 \\ \mu_0 & \mu_1 & \mu_2 & \mu_3 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{array}\right| \div W_2,$$

the determinant itself being defined as $\Delta_3^*(\underline{\theta},\underline{\pi})$. so that

$$\Delta_3(\underline{\theta},\underline{\pi}) = \pi_1^2\pi_2^2(\lambda_1-\lambda_2)^2[\Delta_3^*(\underline{\theta},\underline{\pi}]^2.$$

3 Canonical form for Δ_3^* in the 3pP case

We use column by column reductionism. A multiple of the first column subtracted from the second column reduces the second column to $[2\lambda_1, 2\lambda_2, \lambda_1+\lambda_2]'$, a matrix transpose.

Next using multiples of the first column and reduced second column and subtracting from the third column results in the reduced third column

$$[1+3\theta_1+3\lambda_1+3\lambda_1^2, 1+3\theta_2+3\lambda_2+3\lambda_2^2, 1+3(\theta_1+\theta_2-\mu_1')+\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2]'$$

and for Δ_3^* itself

$$\Delta_3^*(\underline{\theta},\underline{\pi}) = \begin{vmatrix} 1 & 2\lambda_1 & 1+3\mu_1'+6\lambda_1+3\lambda_1^2 \\ 1 & 2\lambda_2 & 1+3\mu_1'+6\lambda_2+3\lambda_2^2 \\ 1 & \lambda_1+\lambda_2 & 1+3(\theta_1+\theta_2-\mu_1')+\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 \end{vmatrix},$$

and a surprising canonical form

$$\Delta_3^*(\underline{\theta},\underline{\pi}) = \begin{vmatrix} 1 & 2\lambda_1 & 3\lambda_1^2 \\ 1 & 2\lambda_2 & 3\lambda_2^2 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 \end{vmatrix},$$

which we describe as a **dichotomized alternant determinant.**

Factoring leads to

$$\Delta_3^*(\underline{\theta},\underline{\pi}) = (\lambda_1-\lambda_2)^3$$

so that for the denominator of the asymptotic covariance determinant we have

$$\Delta_3(\underline{\theta},\underline{\pi}) \sim \pi_1^2\pi_2^2(\lambda_1-\lambda_2)^8 = \pi_1^2\pi_2^2(\theta_1-\theta_2)^8. \tag{6}$$

It is not possible to say what accuracy we have but if $|\theta_1-\theta_2|$ is small we think it should be acceptable (see 2.4.3 in Bowman and Shenton, 2002). From (4), (5) and (6) we have

$$Var_1(\hat{\theta_1}) \sim \frac{\pi_2^2(\lambda_1-\lambda_2)^2}{\Delta_3(\underline{\theta},\underline{\pi})}\begin{vmatrix} 1 & 2\lambda_1 \\ 1 & \lambda_1+\lambda_2 \end{vmatrix}^2 /(\phi_1\phi_2) = \frac{W_3/W_2}{\pi_1^2(\lambda_1-\lambda_2)^4}.$$

Similarly

$$Var_1(\hat{\theta_2}) \sim \frac{W_3/W_2}{\pi_2^2(\lambda_1-\lambda_2)^4}.$$

$$Var_1(\hat{\pi_1}) \sim \frac{4W_3/W_2}{(\lambda_1-\lambda_2)^6}.$$

Table 1. Comparison of 3pP Variances: Theoretical Approximation vs True

θ_1	θ_2	π_1	$Var_1(\hat{\theta_1})$	$Var_1(\hat{\theta_2})$	$Var_1(\hat{\pi_1})$	
1.0	2.0	0.2	822.97	77.30	154.68	Exact
			1103.58	68.97	176.57	Approx.
1.0	6.0	0.2	15.37	11.22	0.34	Exact
			86.09	5.38	0.55	Approx.
5.0	6.0	0.2	29604.26	2067.12	4935.37	Exact
			31675.27	1979.70	5068.04	Approx.
1.0	2.0	0.5	116.81	148.51	125.01	Exact
			140.27	140.27	140.27	Approx.
3.0	4.0	0.5	1262.71	1340.23	1269.85	Exact
			1301.67	1301.67	1301.67	Approx.
5.0	6.0	0.8	4588.84	4713.37	4579.50	Exact
			4624.83	4624.83	4624.83	Approx.
1.0	2.0	0.8	34.75	538.54	82.20	Exact
			33.80	540.88	86.54	Approx.
3.0	4.0	0.8	415.37	6039.41	987.91	Exact
			385.94	6175.05	988.01	Approx.
3.0	9.0	0.8	6.90	112.78	0.40	Exact
			6.64	106.16	0.47	Approx.
5.0	6.0	0.8	1576.69	23325.08	3819.66	Exact
			1494.09	23905.50	3824.88	Approx.
5.0	15.0	0.8	8.38	123.70	0.22	Exact.
			7.17	114.70	0.18	Approx.

Comments: Gratifying support for the basis of the approximant, especially when the difference $|\theta_1 - \theta_2|$ is unity. Even when this difference is large, there are cases in our computations when the approximation is quite good.

4 The five parameter Poisson mixture (5pP)

4.1 The canonical form

The estimators are $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, $\hat{\pi}_1$, and $\hat{\pi}_2$. This extension (3pP to 5pP) adds two additional dimensions, so that we need

$$\sum_{x=0}^{\infty}\left(\frac{x-\theta_i}{\theta_i}\right)\frac{e^{-\theta_i}\theta_i^x}{x!}(x-\mu_1')^s = \nu_s(\theta_i) \qquad (i=1,2,3)$$

and

$$\sum_{x=0}^{\infty}\left(\frac{e^{-\theta_i}\theta_i^x}{x!}-\frac{e^{-\theta_j}\theta_j^x}{x!}\right)(x-\mu_1')^s = \nu_s(i,j) \qquad (i=1,j=3;i=2,j=3)$$

where $s=4$ and $s=5$. For the central moments of a single Poisson component (parameter θ)

$$\mu_5^* = \theta + 10\theta^2, \qquad \mu_6^* = \theta + 25\theta^2 + 15\theta^3.$$

We find,

$$\nu_4(\theta_1) = 1 + 7(2\lambda_1) + 6(3\lambda_1^2) + 6\lambda_1^3 + \mu_1'[10 + 6(2\lambda_1)]$$

and

$$\nu_4^*(1,2) = 1 + 7\mathcal{S}_1(1,2) + 6\mathcal{S}_2(1,2) + \mathcal{S}_3(1,2) + \mu_1'(10 + 6\mathcal{S}_1)$$

where $\mathcal{S}_r(\lambda_i,\lambda_j) = \frac{\lambda_i^{r+1}-\lambda_j^{r+1}}{\lambda_i-\lambda_j}$, $r = 0,1,\cdots$. When these are inserted in the determinant Δ_5^*, the same column reduction process will eliminate all elements excepting $4\lambda_1^3$ and the corresponding $\mathcal{S}_3$.

Similarly

$$\sum_{x=0}^{\infty}\left(\frac{e^{-\theta_i}\theta_i^x}{x!}-\frac{e^{-\theta_j}\theta_j^x}{x!}\right)(x-\mu_1')^5 =$$
$$1 + \underline{15(2\lambda_i)} + \underline{25(3\lambda_i^2)} + 10(4\lambda_i^3) + 5\lambda_i + \mu_1'[25 + 40(2\lambda_i) + 10(3\lambda_i^2)] + 15\mu_1'^2,$$

to a certain extent corresponding to

$$\sum_{x=0}^{\infty}\left(\frac{e^{-\theta_i}\theta_i^x}{x!}-\frac{e^{-\theta_j}\theta_j^x}{x!}\right)(x-\mu_1')^5 = 1 + \underline{15\mathcal{S}_1} + \underline{25\mathcal{S}_2} + 10\mathcal{S}_3 + \mathcal{S}_4 + \mu_1'(25 + 40\mathcal{S}_1 + 10\mathcal{S}_2) + \mu_1'^2.$$

Underscored symbols highlight the correspondence between the two equations. Thus column reduction will eliminate all terms except $5\lambda_1^4$ and $\mathcal{S}_4$. Very surprising!

In general consider

$$
\begin{aligned}
\nu_s &= \sum_{x=0}^{\infty} \frac{e^{-\theta_i}\theta_i^x}{x!}\left(\frac{x-\theta_i}{\theta_i}\right)(x-\mu_1')^s \\
&= \sum_{x=0}^{\infty}(x-\mu_1')^s \frac{\partial}{\partial\theta_i}\frac{e^{-\theta_i}\theta_i^x}{x!} \qquad\qquad (s = 1, 2\cdots) \\
&= \sum_{x=0}^{\infty}\sum_{r=0}^{s}(-1)^r \begin{pmatrix} s \\ r \end{pmatrix} x^{s-r}\mu_1'^r \frac{\partial}{\partial\theta_i}\frac{e^{-\theta_i}\theta_i^x}{x!} \\
&= \sum_{x=0}^{\infty}\sum_{r=0}^{s}(-1)^r \begin{pmatrix} s \\ r \end{pmatrix} \mu_1'^r \{\mathcal{S}_{s-r}^{(1)}x + \mathcal{S}_{s-r}^{(2)}x^{(2)} + \cdots + \mathcal{S}_{s-r}^{(s-r)}x^{(s-r)}\}\frac{\partial}{\partial\theta_i}\frac{e^{-\theta_i}\theta_i^x}{x!} \\
&= \sum_{r=0}^{s}(-1)^r \begin{pmatrix} s \\ r \end{pmatrix} \frac{\partial}{\partial\theta_i}\{\mathcal{S}_{s-r}^{(1)}\theta_i + \mathcal{S}_{s-r}^{(2)}\theta_i^2 + \cdots + \mathcal{S}_{s-r}^{(s-r)}\theta_i^{s-r}\}
\end{aligned}
$$

where $\mathcal{S}_l^{(n)}$ is a Sterling number of the second kind.

<u>Example:</u> $s = 3$

$$
\begin{aligned}
\nu_3 &= \sum_{r=0}^{3}(-1)^r \begin{pmatrix} 3 \\ r \end{pmatrix} \mu_1'^r \frac{\partial}{\partial\theta_i}\left\{\mathcal{S}_{3-r}^{(1)}\theta_i + \mathcal{S}_{3-r}^{(2)}\theta_i^2 + \cdots + \mathcal{S}_{3-r}^{(3-r)}\theta_i^{3-r}\right\} \\
&= \frac{\partial}{\partial\theta_i}\left\{\mathcal{S}_3^{(1)}\theta_i + \mathcal{S}_3^{(2)}\theta_i^2 + \mathcal{S}_3^{(3)}\theta_i^3 - 3\mu_1'(\mathcal{S}_2^{(1)}\theta_i + \mathcal{S}_2^{(2)}\theta_i^2) + 3\mu_1'^2\mathcal{S}_1^{(1)}\theta_i\right\} \\
&= 1 + 3(2\theta_i) + 3\theta_i^2 - 3\mu_1'(1 + 2\theta_i) + 3\mu_1'^2,
\end{aligned}
$$

since $\mathcal{S}_3^{(1)} = 1$, $\mathcal{S}_3^{(2)} = 3$, $\mathcal{S}_3^{(3)} = 1$, $\mathcal{S}_2^{(1)} = 1$, $\mathcal{S}_2^{(2)} = 1$. Note that θ_i^r will always be multiplied by $(r+1)$ as an essential component of the multiplier.

Again, if

$$
\sum_{x=0}^{\infty}\left(\frac{e^{-\theta_i}\theta_i^x}{x!} - \frac{e^{-\theta_j}\theta_j^x}{x!}\right)(x-\mu_1')^s = \nu_s^{**}(i,j)
$$

then expanding $(x-\mu_1')^s$ by the binomial leads to a pattern in $\theta_i + \theta_j$, $\theta_i^2 + \theta_i\theta_j + \theta_j^2$, and so on, in exact correspondence with the pattern displayed for $\nu_r(\theta_i)$. For example,

$$
\begin{aligned}
\frac{\nu_3^{**}}{\theta_i - \theta_j} &= \frac{\theta_i - \theta_j}{\theta_i - \theta_j} + \frac{3(\theta_i^2 - \theta_j^2)}{\theta_i - \theta_j} + \frac{\theta_i^3 - \theta_j^3}{\theta_i - \theta_j} - 3\mu_1'\left(\frac{\theta_i - \theta_j}{\theta_i - \theta_j} + \frac{\theta_i^2 - \theta_j^2}{\theta_i - \theta_j}\right) + 3\mu_1'^2\left(\frac{\theta_i - \theta_j}{\theta_i - \theta_j}\right) \\
&= 1 \cdot 1 + 3(\theta_i + \theta_j) + \theta_i^2 + \theta_i\theta_j + \theta_j^2 - 3\mu_1'(1 + \theta_i + \theta_j) + 3\mu_1'^2,
\end{aligned}
$$

to be compared to

$$
1 + 3(1 + 2\theta_i) + 3\theta_i^2 - 3\mu_1'(1 + 2\theta_i) + 3\mu_1'^2.
$$

4.2 Canonical forms for the covariances

For the asymptotic covariance denominator we have approximately,

$$\Delta_5(\underline{\theta},\underline{\pi}) \sim \frac{\pi_1^2\pi_2^2}{\phi_1\phi_2\phi_3\phi_4\phi_5}\left(\Delta_5^*(\underline{\theta},\underline{\pi})\right)^2,$$

where

$$\Delta_5^* = \left|\begin{array}{ccccc} 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 & 5\lambda_1^4 \\ 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 & 5\lambda_2^4 \\ 1 & 2\lambda_3 & 3\lambda_3^2 & 4\lambda_3^3 & 5\lambda_3^4 \\ \mathcal{S}_1(1,3) & \mathcal{S}_2(1,3) & \mathcal{S}_3(1,3) & \mathcal{S}_4(1,3) & \mathcal{S}_5(1,3) \\ \mathcal{S}_1(2,3) & \mathcal{S}_2(2,3) & \mathcal{S}_3(2,3) & \mathcal{S}_4(2,3) & \mathcal{S}_5(2,3) \end{array}\right| \begin{array}{c} (\theta_1) \\ (\theta_2) \\ (\theta_3) \\ (\pi_1) \\ (\pi_2) \end{array}. \qquad (7)$$

The entries on the extreme right show correspondence between rows and logarithmic derivatives.

Note in particular: (a) the last two rows of the determinant have factors $\lambda_1 - \lambda_3$, and $\lambda_2 - \lambda_3$ respectively. (b) to study covariances refer to (4) and (5), not to minors of (7).

It is evident that (7) has factors $\lambda_1 - \lambda_3$ and $\lambda_2 - \lambda_3$ relating to rows 4 and 5 of the determinant. A closed form can be found by linear operations on rows, such as $\frac{R2-R1}{(2)-(1)}$. This reduction program of linear operators can be used in general provided, for example, Ri and Rj have a factor in common, thus $R_2 - R_4$ is not valid. There may be an optimal reduction strategy.

When more parameters are involved, such as 7pP, 9pP. $\cdots$, a computer factorization package (MAPLE, Mathematika) may be considered.

4.3 Variances and identities for the 5pP case

By two approaches, reductionism and the MAPLE system on factors, we find

$$\Delta_5 \frac{\pi_1^2\pi_2^2}{\phi_1\phi_2\phi_3\phi_4\phi_5}\left\{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)\right\}^8.$$

For $Var_1(\hat{\theta}_1)$, derived from (4) and (5), the approximants are

$$Var_1(\hat{\theta}_1) \sim \frac{W_5/W_4}{\pi_1^2\{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\}^4},$$

the numerator of this variance being

$$\frac{\pi_2^2\pi_3^2}{\phi_1\phi_2\phi_3\phi_4}(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2\begin{vmatrix} 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 \\ 1 & 2\lambda_3 & 3\lambda_3^2 & 4\lambda_3^3 \\ 1 & \mathcal{S}_1(1,3) & \mathcal{S}_2(1,3) & \mathcal{S}_3(1,3) \\ 1 & \mathcal{S}_1(2,3) & \mathcal{S}_2(2,3) & \mathcal{S}_3(2,3) \end{vmatrix}^2 = (\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2\frac{\pi_1^2\pi_2^2\Delta_4^{*2}}{(\phi_1\phi_2\phi_3\phi_4)}$$

where

$$\Delta_4^* = (\lambda_2-\lambda_3)^3(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)$$

producing an identity for the determinant. Thus also

$$Var_1(\hat{\theta}_2) \sim \frac{W_5/W_4}{\pi_2^2\{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\}^4},$$

and

$$Var_1(\hat{\theta}_3) \sim \frac{W_5/W_4}{\pi_3^2\{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\}^4}.$$

For $Var_1(\hat{\pi}_1)$ we consider

$$\begin{vmatrix} 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 \\ 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 \\ 1 & 2\lambda_3 & 3\lambda_3^2 & 4\lambda_3^3 \\ 1 & \mathcal{S}_1(2,3) & \mathcal{S}_2(2,3) & \mathcal{S}_3(2,3) \end{vmatrix} = 2(\lambda_2-\lambda_3)^3(\lambda_1-\lambda_3)(\lambda_2-\lambda_1)(\lambda_2-2\lambda_1+\lambda_3)$$

leading to

$$Var_1(\hat{\pi}_1) \sim \frac{4(\lambda_2-2\lambda_1+\lambda_3)^2(W_5/W_4)}{\{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\}^6},$$

and similarly,

$$Var_1(\hat{\pi}_2) \sim \frac{4(\lambda_1-2\lambda_2+\lambda_3)^2(W_5/W_4)}{\{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\}^6}.$$

Some numerical comparisons are given in Table 2.

Table 2 Approximants and true value for $Var_1(\hat{\theta_1})$, $Var_1(\hat{\theta_2})$, $Var_1(\hat{\theta_3})$, $Var_1(\hat{\pi_1})$, $Var_1(\hat{\pi_2})$.

θ_1	θ_2	θ_3	π_1	π_2		$Var_1(\hat{\theta_1})$	$Var_1(\hat{\theta_2})$	$Var_1(\hat{\theta_3})$	$Var_1(\hat{\pi_1})$	$Var_1(\hat{\pi_2})$
1.0	1.5	2.5	1/3	1/3	Exact	97800	595789	12551	328185	217113
					Approx	176144	891731	11009	556703	396325
					A/E	1.8	1.5	0.88	1.7	1.8
			0.1	0.5	Exact	1223631	304824	10287	373094	244840
					Approx	2224114	450383	8688	632637	450383
					A/E	1.8	1.5	0.84	1.7	1.8
1.0	2.0	2.5	1/3	1/3	Exact	8117	867186	211822	10606	470467
					Approx	12048	975879	192766	14874	433724
					A/E	1.5	1.1	0.9	1.4	0.9
			0.1	0.5	Exact	100332	454024	176029	12078	563404
					Approx	153381	496955	153381	17042	496955
					A/E	1.5	1.1	0.9	1.6	1.7
0.5	0.8	1.5	1/3	1/3	Exact	256831	1258381	16610	2270547	1747291
					Approx	437989	1824195	14776	3655317	2941508
					A/E	1.7	1.5	0.9	1.6	1.7
			0.1	0.5	Exact	3395444	668659	13861	2707559	2081280
					Approx	5735884	955583	12094	4308286	3466966
					A/E	1.7	1.4	0.9	1.6	1.7
0.5	1.0	1.2	1/3	1/3	Exact	75958	13291081	3775517	410677	57042156
					Approx	92822	13929076	3625853	484947	55716304
					A/E	1.2	1.1	1.0	1.2	1.0
			0.1	0.5	Exact	1050209	7446537	3315073	513599	72101446
					Approx	1266379	7601439	3091745	595457	68412955
					A/E	1.2	1.0	0.9	1.2	1.0

Comments: The most important component of the asymptotic variance approximants is W_5/W_4, where W_4 involves central moments up to μ_8, W_5 central moments up to μ_{10}. Note that these associated determinants involve + and - signs, so that accuracy may be a problem.

The comparisons in the table for the ratio (Approx./Exact) are sometimes as large as 1.8 when $\theta_3 = 2.5$, when $\pi_1 = \pi_2 = \pi_3$. However for $\theta_1 = 0.5$, $\theta_2 = 1.0$, $\theta_3 = 1.2$ and varying proportions, the ratios are nearly unity. In conclusion, we may say the tabulations give some support to the underlying basic assumptions.

5 The case of 7pP

5.1 Variances, $Var_1(\hat{\theta}_1)$, $Var_1(\hat{\theta}_2)$, $Var_1(\hat{\theta}_3)$, $Var_1(\hat{\theta}_4)$, $Var_1(\hat{\pi}_1)$, $Var_1(\hat{\pi}_2)$, $Var_1(\hat{\pi}_3)$

The MAPLE and Mathematika system have been used. We found:

$$Var_1(\hat{\theta}_i) \sim \frac{W_7/W_6}{\pi_i^2\{\prod_{j=1}^{4*}(\lambda_i-\lambda_j)\}^4} \qquad (i=1,2,3,4)$$

$$Var_1(\hat{\pi}_3) \sim \frac{4(W_7/W_6)F_7^2(\underline{\lambda})}{\{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)\}^6}$$

where $\prod_i^*$ means a zero term is omitted, and

$$F_7(\underline{\lambda}) = 3\theta_3^2 - 2\theta_3(\theta_1+\theta_2+\theta_4) + \theta_1\theta_2 + \theta_1\theta_4 + \theta_2\theta_4.$$

5.2 Identities for the 7pP

$$\begin{vmatrix} 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 & 5\lambda_2^4 & 6\lambda_2^5 \\ 1 & 2\lambda_3 & 3\lambda_3^2 & 4\lambda_3^3 & 5\lambda_3^4 & 6\lambda_3^5 \\ 1 & 2\lambda_4 & 3\lambda_4^2 & 4\lambda_4^3 & 5\lambda_4^4 & 6\lambda_4^5 \\ 1 & \mathcal{S}_1(1,4) & \mathcal{S}_2(1,4) & \mathcal{S}_3(1,4) & \mathcal{S}_4(1,4) & \mathcal{S}_5(1,4) \\ 1 & \mathcal{S}_1(2,4) & \mathcal{S}_2(2,4) & \mathcal{S}_3(2,4) & \mathcal{S}_4(2,4) & \mathcal{S}_5(2,4) \\ 1 & \mathcal{S}_1(3,4) & \mathcal{S}_2(3,4) & \mathcal{S}_3(3,4) & \mathcal{S}_4(3,4) & \mathcal{S}_5(3,4) \end{vmatrix} \tag{8}$$

$$= (\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)^3(\lambda_2-\lambda_3)^4(\lambda_4-\lambda_3)^3$$

Concerning $\hat{\pi}_3$

$$\begin{vmatrix} 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 & 5\lambda_1^4 & 6\lambda_1^5 \\ 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 & 5\lambda_2^4 & 6\lambda_2^5 \\ 1 & 2\lambda_3 & 3\lambda_3^2 & 4\lambda_3^3 & 5\lambda_3^4 & 6\lambda_3^5 \\ 1 & 2\lambda_4 & 3\lambda_4^2 & 4\lambda_4^3 & 5\lambda_4^4 & 6\lambda_4^5 \\ 1 & \mathcal{S}_1(1,4) & \mathcal{S}_2(1,4) & \mathcal{S}_3(1,4) & \mathcal{S}_4(1,4) & \mathcal{S}_5(1,4) \\ 1 & \mathcal{S}_1(2,4) & \mathcal{S}_2(2,4) & \mathcal{S}_3(2,4) & \mathcal{S}_4(2,4) & \mathcal{S}_5(2,4) \end{vmatrix} \tag{9}$$

$$= 2F_7(\underline{\lambda})(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)(\lambda_1-\lambda_2)^4(\lambda_2-\lambda_4)^3(\lambda_1-\lambda_4)^3$$

Further forms of (8) and (9) may be devised by suitable interchange of subscripts in the λ's and arguments in the $\mathcal{S}$'s.

6 9pP results and other remarks

$$Var_1(\hat{\theta}_i) \sim \frac{W_9/W_8}{\pi_i^2\{\prod_{j=1}^{4*}(\lambda_i - \lambda_j)\}^4}$$

$$Var_1(\hat{\pi}_4) \sim \frac{4(W_9/W_8)F_9^2(\underline{\lambda})}{\{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)\}^6}$$

where

$$\begin{aligned} F_9(\underline{\lambda}) = {} & 4\lambda_4^3 - 3\lambda_4^2(\lambda_1+\lambda_2+\lambda_3+\lambda_5) \\ & + 2\lambda_4(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_5+\lambda_2\lambda_3+\lambda_2\lambda_5+\lambda_3\lambda_5) \\ & - (\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_3\lambda_5+\lambda_2\lambda_3\lambda_5) \end{aligned}$$

From this $Var_1(\hat{\pi}_i)$ $(i = 1, 2, 3)$ may be obtained by interchanges, such as λ_3 for λ_1, λ_4 for λ_3.

Identities

From $Var_1(\hat{\theta}_1)$

$$\left|[\underline{f(\lambda_2)}, \underline{f(\lambda_3)}, \underline{f(\lambda_4)}, \underline{f(\lambda_5)}, \underline{\mathcal{S}(1,5)}, \underline{\mathcal{S}(2,5)}, \underline{\mathcal{S}(3,5)}, \underline{\mathcal{S}(4,5)}]'\right|$$

$$\begin{aligned} = (\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2(\lambda_1-\lambda_5)(\lambda_2-\lambda_3)^4(\lambda_2-\lambda_4)^4 \times \\ (\lambda_2-\lambda_5)^3(\lambda_3-\lambda_4)^4(\lambda_3-\lambda_5)^3(\lambda_4-\lambda_5)^3, \end{aligned}$$

where

$$\begin{aligned} \underline{f(\theta)} &= [1.2\lambda, 3\lambda^2, 4\lambda^3, 5\lambda^4, 6\lambda^5, 7\lambda^6, 8\lambda^7], \\ \underline{\mathcal{S}(i,j)} &= [1, \mathcal{S}_1(i,j), \mathcal{S}_2(i,j), \cdots, \mathcal{S}_7(i,j)] \end{aligned}$$

and $\mathcal{S}_r(i,j)$ is defined in §4.

From $Var_1(\hat{\pi}_4)$

$$\left|[\underline{f(\lambda_1)}, \underline{f(\lambda_2)}, \underline{f(\lambda_3)}, \underline{f(\lambda_4)}, \underline{f(\lambda_5)}, \underline{\mathcal{S}(1,5)}, \underline{\mathcal{S}(2,5)}, \underline{\mathcal{S}(3,5)}]'\right|$$

$$= 2(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)(\lambda_4-\lambda_5)(\lambda_1-\lambda_2)^4(\lambda_1-\lambda_3)^4(\lambda_2-\lambda_3)^4 \times$$
$$(\lambda_1-\lambda_5)^3(\lambda_2-\lambda_5)^3(\lambda_3-\lambda_5)^3 F_9(\underline{\lambda})$$

7 Miscellaneous simple cases

7.1 Some identities

$$\begin{vmatrix} 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 \\ 1 & 2\lambda_2 & 3\lambda_2^2 & 4\lambda_2^3 \\ 1 & 2\lambda_3 & 3\lambda_3^2 & 4\lambda_3^3 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 & \lambda_1^3+\lambda_1^2\lambda_2+\lambda_1\lambda_2^2+\lambda_2^3 \end{vmatrix}$$
$$= (\lambda_1-\lambda_2)^3(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(2\lambda_1+2\lambda_2-4\lambda_3)$$

$$\begin{vmatrix} 1 & 2\lambda_2 & 3\lambda_2^2 \\ 1 & 2\lambda_3 & 3\lambda_3^2 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 \end{vmatrix} = (\lambda_3-\lambda_2)(\lambda_1-\lambda_2)(2\lambda_1+\lambda_2-3\lambda_3)$$

$$\begin{vmatrix} 1 & 2\lambda_1 & 3\lambda_1^2 \\ 1 & 2\lambda_2 & 3\lambda_2^2 \\ 1 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 \end{vmatrix} = (\lambda_1-\lambda_2)^3$$

7.2 Case of 4pP, $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$, $\hat{\pi}_1$, π known

$$Var_1(\hat{\theta}_1) \sim \frac{\phi_4(\lambda_1-\lambda_2)^4(\lambda_2-\lambda_3)^2(2\lambda_1+\lambda_2-3\lambda_3)^2}{\pi_1^2(\lambda_1-\lambda_2)^8(\lambda_2-\lambda_3)^2(\lambda_1-\lambda_3)^2(2\lambda_1+2\lambda_2-4\lambda_3)^2}$$
$$= \frac{(W_4/W_3)(2\lambda_1+\lambda_2-3\lambda_3)^2}{4\pi_1^2(\lambda_1-\lambda_2)^4(\lambda_1-\lambda_3)^2(\lambda_1+\lambda_2-2\lambda_3)^2}$$

Similarly interchange subscripts 1 and 2 for $Var_1(\hat{\theta}_2)$.

$$Var_1(\hat{\theta}_3) \sim \frac{(W_4/W_3)}{4\pi_3^2(\lambda_3-\lambda_1)^4(\lambda_3-\lambda_2)^4(\lambda_1+\lambda_2-2\lambda_3)^2}.$$

7.3 Conjecture

We conjecture the following

$$\begin{vmatrix} 1 & 2\lambda_1 & 3\lambda_1^2 & \cdots & (2s-2)\lambda_1^{2s-3} \\ 1 & 2\lambda_2 & 3\lambda_2^2 & \cdots & (2s-2)\lambda_2^{2s-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2\lambda_s & 3\lambda_s^2 & \cdots & (2s-2)\lambda_s^{2s-3} \\ 1 & \mathcal{S}_1(1,s) & \mathcal{S}_2(1,s) & \cdots & \mathcal{S}_{2s-3}(1,s) \\ 1 & \mathcal{S}_1(2,s) & \mathcal{S}_2(2,s) & \cdots & \mathcal{S}_{2s-3}(2,s) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathcal{S}_1(s-2,s) & \mathcal{S}_2(s-2,s) & \cdots & \mathcal{S}_{2s-3}(s-2,s) \end{vmatrix} = \frac{2A_s^4(\underline{\lambda})F_{2s-1}(\underline{\lambda})}{B_s^3(\underline{\lambda})C_s(\underline{\lambda})} \tag{10}$$

where

$$A_s(\underline{\lambda}) = \prod_{r=1}^{s-1}\prod_{t=r+1}^{s}(\theta_r-\theta_t), \qquad F_{2s-1}(\underline{\lambda}) = \frac{\partial}{\partial\theta_{s-1}}\prod_{r=1}^{s*}(\theta_{2s-1}-\theta_r).$$

($*$ in the product, the zero component is omitted).

$$\begin{aligned} B_s(\underline{\lambda}) &= \prod_{r=1}^{s*}(\theta_{s-1}-\theta_r), \\ C_s(\underline{\lambda}) &= \prod_{r=1}^{s-2}(\theta_r-\theta_s), \qquad (s=3,4,\cdots) \\ &= 1, \qquad (s=2) \end{aligned}$$

The formula is correct for $s = 2,3,4,5$. Interchanges of subscripts is valid provided the same procedure is valid for the basic determinant.

Also let $\Delta_{2s-2}^*(\underline{\lambda})$ be the determnant derived from (10) by delating the first row and adding a new last row $[1, \mathcal{S}_1(s-1,s), \mathcal{S}_2(s-1,s), \cdots, \mathcal{S}_{2s-3}(s-1,s)]$. Then

$$\Delta_{2s-2}^*(\underline{\lambda}) = \frac{\left\{\prod_{r=1}^{s-1}\prod_{t=r+1}^{s}(\theta_r-\theta_t)\right\}^4}{\{\prod_{r=1}^{s*}(\theta_1-\theta_r)\}^2\prod_{r=1}^{s-1}(\theta_r-\theta_s)}$$

$s = 3$

$$\Delta_4^*(\underline{\lambda}) = \frac{(\theta_1-\theta_2)^4(\theta_1-\theta_3)^4(\theta_2-\theta_3)^4}{\{(\theta_1-\theta_2)(\theta_1-\theta_3)\}^2(\theta_1-\theta_3)(\theta_2-\theta_3)} = (\theta_1-\theta_2)^2(\theta_1-\theta_3)(\theta_2-\theta_3)^3$$

8 Binomial mixture

For briefly we consider one case involving 5 parameters, namely

$$P(x,n,\underline{p}) = \sum_{i=1}^{3} \pi_i p_i^x (1-p_i)^{n-x} \qquad (x=0,1,\cdots,n; n=1,2,\cdots; 0<\pi_i<1, 0<p_i<1)$$

when n is known. Maximum likelihood estimators are $n\hat{p}_i$, $\hat{\pi}_i$, for which

$$\sum_{x=0}^{n} (x-\mu_1')^s \frac{\partial}{\partial n p_i} \binom{n}{x} p_i^x (1-p_i)^{n-x} \sim A_{i0}\phi_0 + A_{i1}\phi_1 + \cdots, \qquad (i=1,2,3)$$

and

$$\sum_{x=0}^{n} (x-\mu_1')^s \left\{ \binom{n}{x} p_i^x (1-p_i)^{n-x} - \binom{n}{x} p_j^x (1-p_j)^{n-x} \right\} \sim A_0^{(i,j)}\phi_0 + A_1^{(i,j)}\phi_1 + \cdots$$

$$(i=1, j=3; i=2, j=3)$$

where $\mu_1' = \pi_1 np_1 + \pi_2 np_2 + \pi_3 np_3$, and the expansions refer to orthogonal polynomials with respect to the weight function $P(x,n,\underline{p})$.

The denominator of the associated covariance determinant is

$$\Delta_5 = k_1 \begin{vmatrix} 1 & 2(n-1)p_1 & 3(n-1)^{(2)}rp_1^2 & 4(n-1)^{(3)}p_1^3 & 5(n-1)^{(4)}p_1^4 \\ 1 & 2(n-1)p_2 & 3(n-1)^{(2)}rp_2^2 & 4(n-1)^{(3)}p_2^3 & 5(n-1)^{(4)}p_2^4 \\ 1 & 2(n-1)p_3 & 3(n-1)^{(2)}rp_3^2 & 4(n-1)^{(3)}p_3^3 & 5(n-1)^{(4)}p_3^4 \\ 1 & (n-1)\mathcal{S}_1(1,3) & (n-1)^{(2)}\mathcal{S}_2(1,3) & (n-1)^{(3)}\mathcal{S}_3(1,3) & (n-1)^{(4)}\mathcal{S}_4(1,3) \\ 1 & (n-1)\mathcal{S}_1(2,3) & (n-1)^{(2)}\mathcal{S}_2(2,3) & (n-1)^{(3)}\mathcal{S}_3(2,3) & (n-1)^{(4)}\mathcal{S}_4(2,3) \end{vmatrix}^2$$

where

$$k_1 = -(\pi_1\pi_2\pi_3)^2 (np_1 - np_3)^2 (np_2 - np_3)^2, \quad n > 4.$$

Note that there are common factors $(n-1), (n-1)^{(2)}, \cdots$, in the columns of the determnant.

For the numerator associated with $Var_1(np_1)$ in the covariance determinant we have

$$k_2 \begin{vmatrix} 1 & 2p_2 & 3p_2^2 & 4p_2^3 \\ 1 & 2p_3 & 3p_3^2 & 4p_3^3 \\ 1 & \mathcal{S}_1(1,3) & \mathcal{S}_2(1,3) & \mathcal{S}_3(1,3) \\ 1 & \mathcal{S}_1(2,3) & \mathcal{S}_2(2,3) & \mathcal{S}_3(2,3) \end{vmatrix}^2$$

where

$$k_2 = (\pi_2\pi_3)^2 \left\{(n-1)(n-1)^{(2)}(n-1)^{(3)}\right\}^2 (np_1 - np_3)^2(np_2 - np_3)^2.$$

Hence

$$Var_1(np_1) \sim \frac{W_5/W_4}{\pi_1^2[(n-1)^{(4)}]^2(p_1-p_2)^4(p_1-p_3)^4} \qquad (n > 4)$$

which when $n \to \infty$, $p_i \to 0$. $np_i = \theta_i$, reduces to

$$Var_1(\hat{\theta}_1) \sim \frac{W_5/W_4}{\pi_1^2(\theta_1-\theta_2)^4(\theta_1-\theta_3)^4}$$

as expected. Note that W_4 and W_5 are defined in terms of moments of the binomial mixture random variate; also the form given for $Var_1(np_1)$ has the factor $(n-1)^{(4)}$ appearing, not n^4 as might be surmized.

Proceeding in a similar way, we find

$$Var_1(\hat{\pi}_2) \sim \frac{4(p_1 - 2p_2 + p_3)^2 W_5/W_4}{n^2[(n-1)^{(4)}]^2(p_2-p_1)^6(p_2-p_3)^6}. \qquad (n > 4)$$

Again note the term $(n-1)^{(4)}$ not n^4.

Other binomial mixture, such as 7pB, 9pB, could be considered, doubtless involving canonical alternants.

9 Further remarks

We have studied the asymptotic variances of maximum likelihood estimators for the $(2s-1)$ parameters of the s-component Poisson mixture; there are s $\hat{\theta}$, and $s-1$ proportions. Approximations, using a linked orthogonal systems are set up, these including the form of the asymptotic variances which focuses on the θ-parameters and to a less extent on the proportions.

Since the central moments of the mixture, in the general case, are complicated, an initial attempt at a solution is also complicated, but linear reductionism of determinants involved produces canonical dichotomized alternants. Some interesting identities arise. The factorization program associated with MAPLE (or Mathematika) turns out to be a powerful tool. We are indebted to Robert Byers (Center for Disease Control) for assistance on this direction.

Historically, Thomas Muir, starting out as "Mathematical master in the high school of Glasgow" (around 1882), because Superintendent-General of Education in Cape Colony, and over the course of several years produced five volumes on the History of Determinants for the years 1800 to 1920. There is also a short text book on the subject (A treatise on the theory of Determinants, 1882). Here in §11.4 on alternants he gives the case

$$\begin{vmatrix} \sin x & \cos x & 1 \\ \sin y & \cos y & 1 \\ \sin z & \cos z & 1 \end{vmatrix},$$

and on p.179 the example,

$$\begin{vmatrix} 1 & x_2 + x_3 & x_2 x_3 \\ 1 & x_3 + x_1 & x_3 x_1 \\ 1 & x_1 + x_2 & x_1 x_2 \end{vmatrix} \longleftarrow \zeta(x_1 x_2 x_3)$$

With the onset of digital computers the traditional interest in the theory of determinants as mathematical tools slowly waned.

The reader may refer to "Contributions" to the History of Determinants, Sir Thomas Muir (1920).

An outstanding problem, is the reduction of the component determinants in the maximum likelihood covariance form to canonical dichotomized alternants, the general case being considered. Similarly there is a problem in the simplification of W_s/W_{s-1} involving central moments, of the mixture random variable to order $2s$. Again zero-sum expressions seem to occur such as $(2\lambda_1 + 3\lambda_2 - 5\lambda_3)$ and $3\lambda_3^2 - 2\lambda_3(\lambda_1 + \lambda_2 + \lambda_4) + \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4$.

A cautionary note is needed on sample size, n keeping in mind that $Var_1(\theta)$ refers to the n^{-1} asymptote. Our approximants at least provide the basic structure of variance singularities, throwing quantitative light on the vague notion of "closeness". For example it has been said that if the parameters of the Poisson mixture are close, then large samples may be needed if inferences are required. We have shown that variances of the θ's are inversely related to the square of the corresponding proportions, and to the forth power of the products of the differences $|\theta_i - \theta_j|$; for the proportions a sixth power of the products of the differences is involved. The tabulations provide a guide at least for the 3pP and 5pP.

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