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THE GEOMETRIC DISTRIBUTION'S CENTRAL MOMENTS AND EULERIAN NUMBERS OF THE SECOND KIND

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ABSTRACT

In a previous paper (Shenton and Bowman, 2001) we studied cumulants of the geometric distribution, showing that they lead to Eulerian numbers such as the sets (1,1), (1,4,1), (1,11,11,1). These Eulerian numbers (of the first kind) relate to partitions of s from the factorial $s!$ and as discrete distributions which appear to be nearly the normal distribution when s is large.

Eulerian numbers of the second kind are from the central moments of the geometric distribution and contain sets such as (1,7,1), (1,21,21,1). These are symmetric partitions of $s!E_s$ where E_s refers to the truncated series for the negative exponential e^{-1}. The basic results spring from a finite difference equation of the first order, this being solved by usage of the finite difference operator $(1 - E_{\mathbf{s}}^{-1})^{-1}$.

As with Eulerian numbers of the first kind, there is the property of normality; actually the Eulerian numbers of the second kind, viewed as discrete distribution, are asymptotically normally distributed.

Keyword: Asymptotic moments, asymptotic series, difference-differential equation, maximum likelihood estimator, power series distributions, recurrence relations.

1. INTRODUCTION

In a previous paper (Shenton and Bowman, 2001) we studied the geometric distributions with density

$$G(x;P) = P(1-P)^x \qquad (x = 0, 1, \cdots; 0 < P < 1)$$

and its cumulants $K_s(Q)$, $Q = 1 - P$, and for example,

$$\begin{aligned} K_2 &= Q/P^2 = h^{(1)}(Q)/P^2, \\ K_3 &= (Q + Q^2)/P^3 = h^{(2)}(Q)/P^3, \\ K_4 &= (Q + 4Q^2 + Q^3)/P^4 = h^{(3)}(Q)/P^4, \end{aligned}$$

the coefficients of the polynomials being Eulerian numbers (see Riordan, 1958). A tabulation of the cumulants up to K_{10} is given in Table 3 of Shenton and Bowman (2001). There are several approaches to set up the cumulants using recurrence schemes; one is stated in Eq. (3) of Shenton and Bowman (2001), and another is described in Riordan (1958, p215). There is also the difference-differential equation

$$K_{s+1} = Q\frac{\partial}{\partial Q}K_s. \quad (s = 1, 2, \cdots; K_1 = 1/P).$$

In general,

$$K_s = (h_1^{(s-1)}Q + h_2^{(s-1)}Q^2 + \cdots + h_{s-1}^{(s-1)}Q^{s-1})/P^s, \tag{1}$$

$h_r^{(s)}$ being Eulerian numbers. Since we are going to introduce another similar set, we may refer to $(h_r^{(s)})$ as **Eulerian number of the First Kind**.

In this note we study numbers associated with the central moments $\mu_s(Q)$ of the geometric distribution.

We note at the outset the following simple properties of the discrete Eulerian number distributions, this being defined by the generating function

$$(h_1^{(s)}Q + h_2^{(s)}Q^2 + \cdots + h_{s-1}^{(s)}Q^{s-1})/\sum_{r=1}^{s-1} h_r^{(s)}.$$

- $\sum_{r=1}^{s-1} h_r^{(s)} = s!$
- The polynomial in Eq. (1) is symmetric, with mean $s/2$.
- The skewness is zero and the kurtosis (measure of "peakedness") is $\frac{s(5s-2)}{240}/(\frac{s}{12})^2$ which $\to 3$ as $s \to \infty$. This provides a link with normality for which the kurtosis is 3.

Here we study the central moments of the geometric distribution and set up properties of the associated Eulerian numbers of the Second kind which appear.

2. THE CENTRAL MOMENTS OF THE GEOMETRIC DENSITIES

The moment cumulant relation, subject to existence is

$$\exp\left(\sum_{s=2}^{\infty}\frac{\alpha^s K_s}{s!}\right)=\sum_{s=0}^{\infty}\frac{\mu_s\alpha^s}{s!}. \qquad (\mu_0=1,\mu_1=0) \tag{2}$$

For example,

$$\begin{aligned}
\mu_2 &= K_2,\\
\mu_3 &= K_3,\\
\mu_4 &= K_4+3K_2^2,\\
\mu_5 &= K_5+10K_3K_2,\\
\mu_6 &= K_6+15K_4K_2+10K_3^2+15K_2^3.
\end{aligned}$$

Kendall (1943, p63) gives a list up to K_{10}. A general formula involving partitions is available; differentiation of expression Eq. (2) with respect to α produces a useful recurrence formula. We find

$$\begin{aligned}
\mu_2(Q)=\mu_2 &= Q/P^2 = J^{(2)}(Q)/P^2, \qquad (Q=1-P)\\
\mu_3 &= (Q+Q^2)/P^3 = J^{(3)}(Q)/P^3,\\
\mu_4 &= (Q+7Q^2+Q^3)/P^4 = J^{(4)}(Q)/P^4.
\end{aligned}$$

In general define the sth central moment as

$$\mu_s(Q)=\mu_s=\sum_{r=1}^{s-1} j_r^{(s)}Q^r/P^s, \qquad (0<P<1; s=2,3,\cdots)$$

where $(j_r^{(s)})$ are **Eulerian numbers of the Second Kind**. A list of $(j_r^{(s)})$ is given in Table 1.

Deriving the Eulerian numbers by this approach becomes complicated, so an approach by generating function is used.

Table 1. Eulerian Number of the Second Kind

s	$j_1^{(s)}$	$j_2^{(s)}$	$j_3^{(s)}$	$j_4^{(s)}$	$j_5^{(s)}$	$j_6^{(s)}$	$j_7^{(s)}$	$j_8^{(s)}$	$\cdots$	$\sum$
2	1									1
3	1	1								2
4	1	7	1							9
5	1	21	21	1						44
6	1	51	161	51	1					265
7	1	113	813	813	113	1				1854
8	1	239	3361	7631	3361	239	1			14833
9	1	493	12421	53833	53833	12421	493	1		133496
10	1	1003	42865	320107	607009	320107	42865	1003	1	1334961
11	1	2025	141549	1704693	5494017	5494017	1704693	141549	$\cdots$	14684572
12	1	4071	453905	8422679	42924113	72605303	42924113	8422679	$\cdots$	176214865
13	1	8165	1426803	3947374	302461121	802022261	802022261	302461121	$\cdots$	2290793246
14	1	16355	4423277	178063991	1977056433	7789874691	12172195881	7789874691	$\cdots$	32071105445
15	1	32737	13580674	780900357	12218437871	68767626970	158751895747	158751895747	$\cdots$	481066581676
16	1	65503	41413201	3353014767	72351539361	564446340287	1847023983361	2722620497023	$\cdots$	7697065306817

(The last column gives $J^{(s)} = \sum_{r=1}^{s-1} j_r^{(s)}$. In the interest of space, only half the entries are given with $s > 10$.)

3. A FORMULA FOR THE ELEMENTS IN $\mathbf{j}^{(\mathbf{s})}$, EULERIAN NUMBERS OF THE SECOND KIND

3.1 THE GENERATING FUNCTION FOR CENTRAL MOMENTS OF THE GEOMETRIC DISTRIBUTION

The probability generating function of $G(x;P)$ is

$$E(t^x) = P/(1-Qt),$$

so that the central moment generating function, using $t = e^{\alpha}$, is

$$\frac{Pe^{-\alpha Q/P}}{1-Qe^{\alpha}} = 1 + \frac{\mu_2\alpha^2}{2!} + \frac{\mu_3\alpha^3}{3!} + \cdots,$$

or

$$Pe^{-Q\alpha/P} = \left(1 + \frac{\mu_1\alpha}{1!} + \frac{\mu_2\alpha^2}{2!} + \cdots\right)\left(P - \frac{Q\alpha}{1!} - \frac{Q\alpha^2}{2!} - \cdots\right),$$

where $\mu_0 = 1$, $\mu_1 = 0$. Equating coefficients of $\alpha^r/r!$, we have

$$P\mu_s = \binom{s}{1} Q\mu_{s-1} + \binom{s}{2} Q\mu_{s-2} + \cdots + \binom{s}{s} Q\mu_0 + (-1)^s Q^s. \quad (s = 2, 3, \cdots; \mu_s = J^{(s)}(Q)/P^s).$$

But from Eq. (2) $\mu_s = J^{(s)}(Q)/P^s$, so

$$J^{(s)}(Q) = \binom{s}{1} QJ^{(s-1)}(Q) + \binom{s}{2} QPJ^{(s-2)}(Q) + \cdots + \binom{s}{s} QP^{s-1}J^{(0)}(Q) + (-1)^s Q^s, \qquad (3)$$

where $s = 2, 3, \cdots$, $J^{(0)}(Q) = 1$, $J^{(1)}(Q) = 0$, $J^{(2)}(Q) = Q$. For example,

$$J^{(3)}(Q) = 3QJ^{(2)}(Q) + 3QPJ^{(1)}(Q) + QP^2 - Q^3 = 3Q(Q) + QP^2 - Q^3 = Q(1+Q).$$

The mean value of $J_1^{(s)}$, $J_2^{(s)}, \cdots, J_{s-1}^{(s)}$ is $s/2$.

3.2 SYMMETRY OF THE EULERIAN POLYNOMIALS

Table 1 displays the symmetry up to $J^{(16)}(Q)$. We indicate a mathematical proof which depends on the symmetry of the case of $h_s(Q)$ (Shenton and Bowman, 2001, pp.125-6), Eulerian polynomials of the first kind. In §2 we show the relation between $\mu_s = J^{(s)}(Q)/P^s$, and $K_s = h_{s-1}(Q)/P^s$, central moment and cumulants. For example, for moments in general (Kendall, 1943, p70),

$$\mu_7 = K_7 + 21K_5K_2 + 35K_4K_3 + 105K_3K_2^2,$$

and

$$\mu_r = \sum_{m=0}^{r} \sum \left(\frac{K_1}{p_1}\right)^{\pi_1} \left(\frac{K_2}{p_2}\right)^{\pi_2} \cdots \left(\frac{K_m}{p_m}\right)^{\pi_m} \frac{r!}{\pi_1!\pi_2!\cdots\pi_m!}$$

summed over $\sum_1^m p_s\pi_s \equiv r$ for the second summation; $\pi_1, \pi_2, \cdots$, are non-negative integers.

We know that $P^sK_s(Q)$ is symmetric and that the coefficients are positive (Shenton and Bowman, 2001, 4.3). It follows from the expression for μ_r, the coefficient of $J^{(r)}(Q)$ is positive. Symmetry depends on the symmetry of products such as $h_r(Q)h_s(Q)$. If h_rh_s is symmetric, then all such products are symmetric. Consider the even case. Then

$$h_{2s}(Q) = Q^{(2s-1)/2}(v_{2s-1} + a_1^{(s)}v_{2s-3} + a_3^{(s)}v_{2s-5} + \cdots + a_{2s-3}^{(s)}v_1) \quad (s = 1, 2, \cdots)$$

where $v_s = Q^{-s/2} + Q^{s/2}$, $s = 0, 1, 2, \cdots$, and $v_s = 0$ if $s < 1$.

But v_sv_r is clearly symmetric, $r = 0, 1, \cdots$. A similar treatment applies to the odd case. Hence returning to μ_rP^r, the Eulerian polynomials of the second kind have positive integer coefficients and are symmetric. Hence all odd moments referred to the mean are 0, the mean for $J^{(s)}(Q) = s/2$.

3.3 THE SUM $\sum_{\mathbf{r=0}}^{\mathbf{s-1}} \mathbf{j_r^{(s)}}, (\mathbf{r = 2, 3, \cdots})$

Insert $Q = 1$, $P = 0$, in Eq. (3). Then

$$J^{(s)}(1) = sJ^{(s-1)}(1) + (-1)^s. \quad (s = 2, 3, \cdots) \tag{4}$$

Now let $J^{(s)}(1) = s!H^{(s)}(1)$. Then

$$H^{(s)}(1) = H^{(s-1)}(1) + (-1)^s/s!;$$

leading to

$$J^{(s)}(1) = s!\left(1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^s}{s!}\right)$$

which we define as $s!E_s$ or $J^{(s)}(1)$ where $E_0 = 1$, $E_1 = 0$, $E_2 = 1/2$, $E_3 = 1/3$, $E_4 = 9/24$, etc.

Examples of this sum, $J^{(s)}(1)$, are given in Table 1. Eq. (4) readily sets up whatever values are required.

Eulerian discrete distributions are now derived, and from $J^{(s)}(0)$, the probabilities are

$$j_1^{(s)}/J^{(s)}(1),\ \ j_2^{(s)}/J_1^{(s)}, \cdots j_r^{(s)}/J^{(s)}(1), \cdots, j_{s-1}^{(s)}/J^{(s)}(1) \qquad (s = 2, 3, \cdots).$$

Using factorial moments in this case involves heavy algebra so specialized that MAPLE and other languages are of little help. Factorial moment methods are more efficient in the discrete case than central moments; note the cases of the binomial, and Poisson for which the factorial moment generating functions are $(1 + p\alpha)^n$ and $\exp(\theta\alpha)$ respectively. However, the advantage fails somewhat when sampling moments are concerned. We therefore turned to the use of central moment generating functions.

4. CENTRAL MOMENT GENERATING FUNCTION

4.1 THE FORMULAS

The basic result follows from the modification of equation Eq. (3); here set $Q = e^\alpha$, $P = 1 - e^\alpha$, noting that the mean of the associated random variable for the sth case if $s/2$. Let

$$C^{(s)}(\alpha) = e^{-s\alpha/2} J^{(s)}(Q)/(s!E_s). \qquad (Q = e^\alpha)$$

For example,

$$\begin{aligned}
s = 2:&\ \ C^{(2)}(\alpha) = e^{\alpha/2} \qquad (2!E_2 = 1)\\
s = 3:&\ \ C^{(3)}(\alpha) = (e^{-\alpha} + e^{\alpha})/2 \qquad (3!E_3 = 2)\\
s = 4:&\ \ C^{(4)}(\alpha) = (e^{-\alpha} + 7 + e^{\alpha})/9 \qquad (4!E_s = 9)\\
s = 5:&\ \ C^{(5)}(\alpha) = (e^{-3\alpha/2} + 21e^{-\alpha} + 21e^{\alpha} + e^{3\alpha/2})/44. \qquad (5!E_5 = 44)
\end{aligned}$$

Note that in the numerators, when $\alpha = 0$, the sum is $s!E_s$.

From Eq. (3) we derive the finite difference equation for the moment generating function, namely

$$\begin{aligned}
C^{(s)}(\alpha) = {}& e^{\alpha/2} C^{(s-1)}(\alpha) \frac{E_{s-1}}{E_s} + \frac{(1-e^\alpha)}{2!} C^{(s-2)}(\alpha) \frac{E_{s-2}}{E_s} + \frac{e^{-\alpha/2}(1-e^\alpha)^2}{3!} C^{(s-3)}(\alpha) \frac{E_{s-3}}{E_s}\\
&+ \cdots + \frac{e^{-(s-2)\alpha/2}(1-e^\alpha)^{s-1}}{s!} C^{(0)}(\alpha) \frac{E_0}{E_s} + \frac{(-1)^s e^{s\alpha/2}}{s!E_s}, \qquad (s = 2, 3, \cdots)
\end{aligned}$$

leading to

$$\begin{aligned}
E_s C^{(s)}(\alpha) = {}& e^{\alpha/2} E_{s-1} C^{(s-1)}(\alpha) + \frac{(1-e^\alpha)}{2!} E_{s-2} C^{(s-2)}(\alpha) + \frac{e^{-\alpha/2}(1-e^\alpha)^2}{3!} E_{s-3} C^{(s-3)}(\alpha)\\
&+ \cdots + \frac{e^{-(s-2)\alpha/2}(1-e^\alpha)^{s-1}}{s!} E_0 C^{(0)}(\alpha) + \frac{(-1)^s e^{s\alpha/2}}{s!} \qquad (s = 2, 3, \cdots). \qquad (5)
\end{aligned}$$

4.2 ELEMENTARY ALGEBRA OF $\mathbf{E_s}$

First of all,

$$E_s - E_{s-1} = \frac{(-1)^s}{s!}$$

and

$$(1 - E_{\mathbf{s}}^{-1})\frac{(-1)^s}{s!} = (-1)^s \left\{ \frac{1}{s!} - \frac{1}{(s-1)!} + \frac{1}{(s-2)!} + \cdots + \frac{(-1)^s}{0!} \right\} = E_s.$$

There is the fundamental identity

$$sE_s = (s-1)E_{s-1} + E_{s-2} \qquad (s = 2, 3, \cdots)$$

or

$$(s-1)E_{s-1} = sE_s - E_{s-2}. \tag{6}$$

Summation

$$\sum_{r=0}^{s} E_r = (1 - E_{\mathbf{s}}^{-1})^{-1} E_s = (s+2)E_{s+2}. \tag{7}$$

4.3 THE VARIANCE OF THE DISCRETE EULERIAN RANDOM VARIATES

Differentiating Eq. (5) twice with respect to α, and setting $\alpha = 0$, we have for the variance $C_2^{(s)}$ the difference equation

$$E_s C_2^{(s)} = E_{s-1} C_2^{(s-1)} + \frac{E_{s-1}}{4} - \frac{E_{s-2}}{2} + \frac{E_{s-3}}{3} + \frac{(-1)^s}{4} \left\{ \frac{1}{(s-2)!} + \frac{1}{(s-1)!} \right\}$$

and the last term on the right-hand side is $\frac{(E_{s-2}-E_{s-3})}{4} - \frac{(E_{s-1}-E_{s-2})}{4}$. Here,

$$E_s C_2^{(s)} = E_{s-1} C_2^{(s-1)} + \frac{E_{s-3}}{12}.$$

Replace s by r and sum over $r = 1, 2, t \cdot s$, yielding from Eq. (7)

$$\begin{aligned} E_s C_2^{(s)} &= [(s-2)E_{s-3} + E_{s-4}]/12 = [(s-3)E_{s-3} + E_{s-3} + E_{s-4}]/12 \\ &= (s-1)E_{s-1}/12 = [sE_s - E_{s-2}]/12. \end{aligned}$$

Hence,

$$C_2^{(s)} = \frac{sE_s - E_{s-2}}{12E_s} = \frac{(s-1)E_{s-1}}{12E_s} \qquad (s = 2, 3, \cdots)$$

with asymptotic

$$\hat{\nu}_2^{(s)} \sim \frac{s-1}{12}. \qquad (s \to \infty)$$

Note that

$$\hat{\nu}_2^{(s)} - \nu_2^{(s)} \sim \frac{(-1)^s}{12} \left(\frac{1}{s!} - \frac{1}{(s-1)!} \right). \qquad (s \to \infty)$$

In general, when power series are involved, we would expect an asymptote of the form

$$\hat{\nu}_2^{(s)} \sim \frac{s}{12} - \frac{1}{12} + \frac{\alpha}{s} + \frac{\beta}{s^2} \cdots$$

for example.

4.4 THE FOURTH CENTRAL MOMENTS $\nu_4^{(s)}$

From Eq. (5)

$$\frac{\partial^4 E_s C_\alpha^{(s)}}{\partial \alpha^4} |_{\alpha=0} = E_{s-1} C^{(s)} + W_s(2) + W_s(1) + W_s(0),$$

where

$$\begin{aligned}
W_s(2) &= \omega_{s-2}/8 - \omega_{s-3}/4 + \omega_{s-4}/6, \\
W_s(1) &= \frac{E_{s-1}}{16} - \frac{E_{s-2}}{2} + \frac{5E_{s-3}}{6} - \frac{E_{s-4}}{2} + \frac{E_{s-5}}{5}, \\
W_s(0) &= \frac{(-1)^s s^4}{16 s!} = \frac{(-1)^s}{16}\left\{\frac{1}{(s-1)!} + \frac{7}{(s-2)!} + \frac{6}{(s-3)!} + \frac{1}{(s-4)!}\right\}.
\end{aligned}$$

Using Eq. (6) $W_s(2)$ may be simplified. Thus

$$\begin{aligned}
W_s(2) &= \omega_{s-2}/8 - \omega_{s-3}/4 + (\omega_{s-3} - E_{s-5})/6 \qquad (\omega_s = sE_s) \\
&= \omega_{s-2}/8 - \omega_{s-3}/12 - E_{s-5}/6 \\
&= \omega_{s-2}/24 + E_{s-4}/12 - E_{s-5}/6 \\
&= \omega_{s-1}/24 - E_{s-3}/24 + E_{s-4}/12 - E_{s-5}/6 \\
&= \omega_s/24 - E_{s-2}/24 - E_{s-3}/24 + E_{s-4}/12 - E_{s-5}/6.
\end{aligned}$$

For the sum

$$\begin{aligned}
(1 - E_{\mathbf{s}}^{-1})^{-1} W_s(2) &= \frac{s(s+1)E_s}{48} - \frac{sE_s}{24} - \frac{(s-1)E_{s-1}}{24} + \frac{(s-2)E_{s-2}}{12} - \frac{(s-3)E_{s-3}}{6} \\
&= \frac{s(s+1)E_s}{48} - \frac{E_{s-2}}{48} - \frac{sE_s}{6} + \frac{E_{s-2}}{8} + \frac{E_{s-3}}{12} + \frac{E_{s-4}}{6}. \qquad (8)
\end{aligned}$$

Similarly,

$$(1 - E_{\mathbf{s}}^{-1})^{-1} W_s(1) = \frac{(s+1)E_{s+1}}{16} - \frac{sE_s}{2} + \frac{8sE_s}{15} - \frac{8E_{s-2}}{15} + \frac{3E_{s-3}}{10} - \frac{E_{s-4}}{5}, \qquad (9)$$

and

$$(1 - E_{\mathbf{s}}^{-1})^{-1} W_s(0) = (-E_{s-1} + 7E_{s-2} - 6E_{s-3} + E_{s-4})/16. \qquad (10)$$

Collecting terms in Eqs.(8-10) yields the final exact fourth central moment of the Eulerian discrete distribution.

$$\nu_4^{(s)} = \frac{s^2}{48} - \frac{s}{20} + \frac{1}{E_s}\left\{\frac{E_{s-2}}{120} + \frac{E_{s-3}}{120} + \frac{7E_{s-4}}{240}\right\} \qquad (s = 2, 3, \cdots)$$

and the asymptotic

$$\hat{\nu}_4^{(s)} = \frac{s^2}{48} - \frac{s}{20} + \frac{11}{240}, \qquad (s \to \infty)$$

with error

$$\begin{aligned}\hat{\nu}_4^{(s)} - \nu_4^{(s)} &= \frac{11}{240} - \left(\frac{E_{s-2}}{120} + \frac{E_{s-3}}{120} + \frac{7E_{s-4}}{240}\right)/E_s \\ &= \frac{E_s - E_{s-2}}{120} + \frac{E_s - E_{s-3}}{120} + \frac{7(E_s - E_{s-4})}{240} \qquad (s = 3, 4, \cdots)\end{aligned}$$

and

$$\begin{aligned}E_s - E_{s-2} &= (-1)^s\left(\frac{1}{s!} - \frac{1}{(s-1)!}\right), \\ E_s - E_{s-3} &= (-1)^s\left(\frac{1}{s!} - \frac{1}{(s-1)!} + \frac{1}{(s-2)!}\right), \\ E_s - E_{s-4} &= (-1)^s\left(\frac{1}{s!} - \frac{1}{(s-1)!} + \frac{1}{(s-2)!} - \frac{1}{(s-3)!}\right).\end{aligned}$$

5. ASYMPTOTIC NORMALITY

We proceed by induction. Assume

$$\nu_{2r}^{(s)} = [1 \cdot 3 \cdot 5 \cdots (2r-1)]s^r/12^r \qquad (s \to \infty).$$

Differentiate Eq. (3) $2r+2$ times and set $\alpha = 0$. Then

$$\begin{aligned}\nu_{2r}^{(s)} &\sim \binom{2r+2}{2}\left\{\frac{\nu_{2r}^{(s-1)}}{4} - \frac{\nu_{2r}^{(s-2)}}{2} + \frac{\nu_{2r}^{(s-3)}}{3}\right\} \\ &\sim \frac{(r+1)(2r+1)}{12}\left\{\frac{1 \cdot 3 \cdot 5 \cdots (2r-1)s^r}{12^r}\right\} \\ &= \left\{\frac{1 \cdot 3 \cdot 5 \cdots (2r+1)s^r}{12^{r+1}}\right\}(r+1).\end{aligned}$$

Hence summing, the result is universally true, and the distribution of Eulerian numbers of the second kind is asymptotically normal.

A tabulation of means and higher moments is given in Table 2.

Table 2. Mean, Variance, Kurtosis for the
Eulerian Second Kind Discrete Distribution

s	Mean	Variance		μ_4	Kurtosis
4	2	2/9	0.222222222	2/9	9/2
5	5/2	15/44	0.340909091	51/176	187/75
6	3	110/265	0.415094340	0.505660377	2.934710742
7	7/2	927.5/1854	0.500269687	0.714064186	2.853178050
8	4	8652/14833	0.583294007	0.979707409	2.879528388
9	9/2	88998/133496	0.666671661	1.283240696	2.887248310
10	5	1001220/1334961	0.749999438	1.629180178	2.896324656
11	11/2		0.833333390	2.016664950	2.903997132
12	6		0.916666662	2.445833527	2.910744065
13	13/2		1.000000000	2.916666647	2.916666647
14	7		1.083333333	3.429166668	2.921893495
15	15/2		1.166666667	3.983333333	2.926530610
16	8		1.250000000	4.579166667	2.930666667
18	9		1.416666667	5.895833333	2.937716262
20	10		1.583333333	7.379166667	2.943490307

$$\text{mean} = s/2, \qquad \text{variance} = (s-1)E_{s-1}/(12E_s),$$

$$\mu_3 = 0,$$

$$\mu_4 = 11/240 - s/20 + s^2/48 + (E_{s-2}/120 + E_{s-3}/120 + 7E_{s-4}/240)/E_s,$$

$$\beta_2 = (3 - \frac{36}{5s} + \frac{33}{5s^2})/(1-s)^2 = 3 - \frac{6/5}{s} + \frac{6/5}{s^2} + \frac{18/5}{s^3} + \frac{6}{s^4} + \frac{42/5}{s^5} - \frac{51/5}{s^6} + \frac{198/5}{s^7} \cdots.$$

6. MONTAGE OF e'S

6.1 CONTINUED FRACTIONS

The exponentials e and e^{-1} have attracted considerable interest historically, the chief contributors being Euler and Gauss; the series aspect is considered and transformation into rational function sequences set up. A pivotal result is due to Gauss and the hypergeometric series (in various forms) $1 + \frac{a \cdot b \cdot x}{c1!} + \frac{a(a+1)b(b+1)x^2}{c(c+1)2!} \cdots$. For the usual ladder form, see Wall (1948). We use notation

$$\frac{a}{1+}\ \frac{b}{1+}\ \frac{c}{1+}\ \frac{d}{1+} \cdots.$$

We offer some results.

$$e^{-1} = \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{3+}\ \frac{1}{2+}\ \frac{1}{5+}\ \frac{1}{2+}\cdots \qquad \text{(Wall, 1948; Gauss)}$$

$$\begin{aligned} e^{-1} - E_{s-1} &= \frac{(-1)^s}{s!}\left\{\frac{1}{1+}\ \frac{1}{s+1-}\ \frac{1}{s+2+}\ \frac{s+1}{s+3-}\ \frac{2}{s+4+}\ \frac{s+3}{s+5-}\ \frac{3}{s+6+}\cdots\right\} \\ &= \frac{(-1)^s}{s!}\Phi(1, s+1, -1), \end{aligned}$$

where

$$\Phi(b,c,z) = 1 + \frac{bz}{c} + \frac{b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots \qquad \text{(Wall, 1948, Gauss)}$$

$$\begin{aligned} e^{-1} &= 1 - \frac{2}{3+}\ \frac{1}{6+}\ \frac{1}{10+}\ \frac{1}{14+}\ \frac{1}{18+}\cdots \qquad \text{(H.M.F., 4.2.40, (1964))} \\ e &= 2 + \frac{1}{1+}\ \frac{1}{2+}\ \frac{2}{2+}\ \frac{3}{4+}\ \frac{4}{5+}\cdots \qquad \text{(Perron; Euler, p19)} \end{aligned}$$

$$\frac{E_s}{E_{s+1}} = \frac{s+1}{s+}\ \frac{s}{s-1+}\ \frac{s-1}{s-2+\cdots+}\ \frac{3}{2}.$$

6.2 A DISTRIBUTION ASSOCIATED WITH $\mathbf{E_s}$ AND ASYMPTOTIC UNIFORMITY

We consider the random variable $\hat{r}$ with probability function

$$Pr(\hat{r} = r) = E_r/[(s+2)E_{s+2}]. \qquad (s = 2, 3, \cdots)$$

Then using the identities

$$\sum_{r=0}^{s} r^{(\lambda)}E_r = [(s+1)^{(\lambda+1)}E_s + (-1)^s E_{s-\lambda-1}]/(s+1) \qquad (s = \lambda+1, \lambda+2, \cdots; \lambda = 0, 1, 2, \cdots)$$

the factorial moments of $\hat{r}$ may be set up. For an example,

$$\begin{aligned} \text{Mean}: \quad & \hat{\nu}_1^{(s)} = \frac{s}{2} + \frac{(s-1)E_{s-1}}{6sE_s} \sim \frac{s}{2} + \frac{1}{6} \qquad (s \to \infty) \\ \text{Variance}: \quad & \hat{\nu}_2^{(s)} = \lambda_s + \frac{4\lambda_s^2}{s^2} \qquad (\lambda_s = (s-1)E_{s-1}/(12E_s)) \\ \text{Skewness}: \quad & \hat{\nu}_3^{(s)} = \frac{s}{24} - \frac{6\lambda_s^2}{s} - \frac{1}{10} + \frac{2}{s}f(E_s) + \frac{16\lambda_s^3}{s^3} \qquad (s = 2, 3, \cdots) \end{aligned}$$

and

$$f(E_s) = \left\{\frac{E_{s-2}}{120} + \frac{E_{s-3}}{120} + \frac{7E_{s-4}}{240}\right\}/E_s.$$

Omitting details we find when s is large that the mth non-central moment is $s^m/(m+1)$. Thus for the variance, $\sigma^2 = s^2\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{s^2}{12}$,

$$\text{Skewness}: \quad \hat{\nu}_3^{(s)} \sim s^3\left(\frac{1}{4} - \frac{3\cdot 1}{2\cdot 3} + \frac{2}{8}\right) = 0$$
$$\text{Kurtosis}: \quad \beta_2 \sim s^4\left(\frac{1}{5} - \frac{4\cdot 1}{2\cdot 4} + \frac{6\cdot 1}{4\cdot 3} - \frac{3}{16}\right)/(s^2/12)^2 = 1.8$$

Here, as far as the moments are concerned, the random variable $\hat{r}/s$ is distributed as a uniform density $U(0,1)$.

7. SOME CONCLUDING REMARKS

The Eulerian polynomials $(h_s(Q))$ of the first kind relate to partitions of size s from $s!$. An unusual recurrence appears from the cumulants of the geometric distribution, and $\ln(1/Q)$ plays a significant role. Strangely enough, the geometric distribution has probability function $(1-q)\exp(x\ln q)$. For Eulerian polynomials of the second kind, the link is with the truncated negative exponential e^{-1}. E_s is used to denote a truncated form. Here the associated central moment generating function is recursively defined, and moments arise from derivatives; the finite difference operator E plays a fundamental role in the solution used as $(1-E_{\mathbf{s}}^{-1})^{-1}$. Properties of E_s are given.

In our study of Eulerian discrete distributions, asymptotic normality seems to be a characteristic. We have also looked at convolutions of these distributions and also derivative sets of distributions with moment generating functions $\frac{\partial}{\partial\alpha}\left\{\frac{2}{s}e^{s\alpha/2}C^{(s)}(\alpha)\right\}$. This family is also asymptotically normally distributed from a moment point of view.

An introduction to the interface between finite difference calculus and probability is given in Riodan (1958).

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