Far East J.Theo.Stat. 5(1)(2001),113-142

The Geometric Distribution, Cumulants, Eulerian Numbers, and the Logarithmic Function

L.R. Shenton
Department of Statistics
University of Georgia
Athens, Georgia 30602

K.O. Bowman*
Computer Science and Mathematics Division
Oak Ridge National Laboratory
P.O.Box 2008, MS-6367
Oak Ridge, Tennessee 37831-6367
bowmanko@ornl.gov

February 12, 2001

Abstract

A recurrence relation is given for the cumulants, this being linear in the cumulants. The cumulants involve Eulerian numbers this also being true for the cumulants of the binomial and Fisher's logarithmic series distributions. An approximation to the cumulants of the geometric distribution involves a logarithmic function which in turn leads to approximants to $-[\ln(1-X)]/X$. Maximum likelihood estimator is considered, asymptotic moments derived and compared with our previous studies (Bowman and Shenton, 1998, 1999). Exact expressions, using continued fractions, are given for the moments of the geometric random variate.

Keyword: Asymptotic moments, asymptotic series, difference-differential equation, maximum likelihood estimator, power series distributions, recurrence relations.

1 Introduction

The geometric distribution is defined by the probability function

$$G(x, P) = P(1-P)^x. \qquad (x = 0, 1, 2, \cdots; 0 < P < 1) \tag{1}$$

The study was motivated by the need to find simple non-trivial cases of maximum likelihood estimation for which exact moments (bias and first order bias, variance and 2nd order variance, dominant skewness and dominant kurtosis, the latter being measured by moment ratios) were known. Obvious examples relate to the binomial, normal and one form of the gamma distribution. We needed further examples in order to verify our asymptotic formulas for low order moments (means, variances, etc.) in the simultaneous estimate of one or more parameters (Bowman and Shenton, 1998, 1999, 2000, 2001; Bowman and Williams, 2000).

It was pure-serendipity that we looked at (1), the geometric distribution involving one parameter. Our general formulae for moments of maximum likelihood estimators are quite complicated, but the one parameter case is more user friendly. Now the maximum likelihood estimator $\hat{P}$ of P is given by

$$\hat{P} = \frac{1}{1+\bar{x}} \tag{2}$$

or

$$\hat{P} - P = -P^2 y/(1 + Py)$$

where for a random sample $(x_1, x_2, \cdots, x_N)$ the mean is $\bar{x}$, and $y = \bar{x} - \mu_1'(x) = \bar{x} - Q/P$. If we know the moments of the mean $\bar{x}$, then from (2) we can set up low-order moments of $\hat{P}$, using correction formulas such as

$$\begin{aligned} Var(\hat{P}) &= E(\hat{P} - P)^2 - [E(\hat{P} - P)]^2, \\ \mu_3(\hat{P}) &= E(\hat{P} - P)^3 - 3[E(\hat{P} - P)^2]E(\hat{P} - P) + 2[E(\hat{P} - P)]^3. \end{aligned}$$

It seemed to us that setting up cumulants for the geometric distribution random variable might be the most promising choice, for from these, cumulants of the mean are readily derived; for $K_s(\bar{x}) = K_s(x)/N^{s-1}$. We found a recursive formula for $K_s(x)$, from which an approximation scheme readily followed. These results were of a different kind from those given by Johnson, Kotz, and Kemp (1993, p.108) namely

$$K_{r+1} = Q\frac{\partial K_r}{\partial Q}. \qquad (r \geq 1)$$

In the new results the pivotal entity is $\ln(1/Q)$.

Now, we have defined the polynomials $h_s(Q)$ as

$$P^{s+1}K_{s+1} = h_s(Q)$$

where $h_s(Q)$ turns out to be of degree s, each coefficient being a positive integer and the set of coefficients being a partition of s from $s!$. Approximations to $[\ln(1-x)]/x$ are derived and also the structure of the components in $h_s(Q)$.

In the interests of space, we shall merely quote results indicating appropriate references.

This paper may be regarded as a Binomial montage from $(1+x)^n$, n a positive integer, n negative, to $(pt+q)^n$ and finally $(1+P-Pt)^{-k}$, and in the latter $k>0$, $P>0$.

2 Low Order Moments

It is assumed throughout that $0<P<1$, that $P+Q=1$, and that the symbol $\sim$ represents an asymptotic result for which in general N, the sample size, tends to ∞.

2.1 Moments of the Geometric Random Variate

$$\begin{aligned}
&\mu_1'(x) = Q/P, \qquad\qquad \mu_2(x) = Var(x) = Q/P^2,\\
&\mu_3(x) = Q(Q+1)/P^3, \qquad \sqrt{\beta_1(x)} = (Q+1)/\sqrt{Q},\\
&\mu_4(x) = 3[\mu_2(x)]^2 + (Q/P^4)(Q^2+4Q+1),\\
&\beta_2(x) = \mu_4(x)/\mu_2^2(x) = 3 + (Q^2+4Q+1)/Q.
\end{aligned}$$

Comments: The skewness, measured by a moment ratio $\sqrt{\beta_1}$ is large if $Q(>0)$ is small.

2.2 Asymptotic Moments of the Maximum Likelihood Estimator $\hat{P}$ of P

Mean:

$$E(\hat{P}) \sim P + PQ/N + PQ(2Q-1)/N^2 + PQ(6Q^2-6Q+1)/N^3$$

Variance:

$$\mu_2(\hat{P}) \sim QP^2/N + QP^2(6Q-2)/N^2 + QP^2(34Q^2-26Q+3)/N^3$$

$$\mu_3(\hat{P}) \sim QP^3(5Q-1)/N^2 + QP^3(68Q^2-39Q+3)/N^3$$

Skewness:

$$\sqrt{\beta_1(\hat{P})} \sim (5Q-1)/\sqrt{NQ}.$$

Kurtosis

$$\mu_4(\hat{P}) \sim 3Q^2P^4/N^2 + QP^4(85Q^2-32Q+1)/N^3,$$

$$\begin{aligned}
\beta_2(\hat{P}) &\sim (\mu_{42}/N^2 + \mu_{43}/N^3)/(\mu_{21}/N + \mu_{22}/N^2)^2\\
&\sim 3 + \left(\frac{1}{Q} - 20 + 49Q\right)/N.
\end{aligned}$$

Comment: The series may diverge. The results for the mean to the N^{-2} term, variance to the N^{-2} term, skewness to the $N^{-1/2}$ term and dominant kurtosis $1/Q - 20 + 49Q$ agree with those described in Shenton and Bowman (1977), Bowman and Shenton (1998,1999). The verification for the formulas in the case of the two parameter gamma density $e^{-x/a}(x/a)^\rho/a\Gamma(\rho)$ is given in Bowman and Shenton (1999, p2502).

3 Exact Moments of the Maximum Likelihood Estimator of P_N

3.1 Integrals of $\hat{P}_N$

For the geometric random variables, the probability generating function is

$$Et^x = P/(1-Qt) \tag{3}$$

so that for $s(N) = N\bar{x}$,

$$Et^{s(N)} = P^N/(1-Qt)^N$$

and by integrating the expression $t^{s(N)+N-1}(\ln t)^{r-1}$ we have

$$E(\hat{P}_N)^r = \frac{N^r P^N}{(r-1)!}\int_0^1 \frac{t^{N-1}[\ln(1/t)]^{r-1}dt}{(1-Qt)^N} \qquad (r = 1, 2, \cdots) \tag{4}$$

and $\mu_2(\hat{P}_N)$, $\mu_3(\hat{P}_N)$, etc. may be evaluated using the usual correction formulas, such as

$$\mu_2(\hat{P}) = Var(\hat{P}) = \mu_2'(\hat{P}) - [\mu_1'(\hat{P})]^2 = E(\hat{P}^2) - [E(\hat{P})]^2,$$

$$\mu_3(\hat{P}) = E(\hat{P}^3) - 3[E(\hat{P})^2]E(\hat{P}) + 2[E(\hat{P})]^3.$$

The formula in (4) for $E(\hat{P}_N)^r$ reduces to a case for which the integrand may have for numerator the component

$$[1-(1-Qt)]^{N-1}/Q^{N-1}$$

leading to the formula

$$\frac{E(\hat{P}_N)}{N} = (-1)^{N-1}\left(\frac{P}{Q}\right)^N \left\{\ln(1+Q/P) - \frac{Q/P}{1} + \frac{(Q/P)^2}{2} - \frac{(Q/P)^3}{3} + \cdots + (-1)^{N-1}\frac{(Q/P)^{N-1}}{N-1}\right\}, \tag{5}$$

for $N = 2, 3, \cdots$, $0 < P < 1$, $P + Q = 1$. Clearly the term on the right is in the form of a residue. For $N = 1$,

$$E(\hat{P}_1) = (P/Q)\ln(1+Q/P).$$

3.2 Infinite Series and $E(\hat{P}^r)$

From the probability generating function in (1) and $\hat{P} = N/[N + s(N)]$ we have the series

$$E(\hat{P}^r) = N^r \sum_{s=0}^{\infty} \frac{\Gamma(N+s)P^N Q^s}{\Gamma(N)(N+s)^r s!}. \qquad (r = 1, 2, \cdots)$$

Setting a stopping rule for the series, non-central moment of $\hat{P}$ may be set up and converted to central moments.

3.3 Rational Fraction (Padé) Sequences for $E(\hat{P})$

3.3.1 The Mean Value of $\hat{P}$ and Stiltjies

A special case is

$$F(z) = \int_0^\infty \frac{d\sigma(t)}{z+t} = N \sum_{s=0}^{\infty} \frac{\Gamma(N+s)P^N Q^s}{\Gamma(N)(z+s)s!}, \qquad (\Re(z) > 0)$$

where $\sigma(\cdot)$ is a step function, bounded, and with infinitely many points of increase (Shohat and Tamarkin (1943), Wall (1948), Bowman and Shenton (1989) and Brezinski (1978, 1980a, 1980b)).

In a number of cases, the corresponding continued fraction may be set up quickly if we knew the orthogonal system associated with $\sigma(\cdot)$). In the present case these orthogonal polynomials have been described by Aitken and Gonin (1934). First of all we set up a J-fraction of the form

$$\frac{b_0}{z + a_1 -} \; \frac{b_1}{z + a_2 -} \; \frac{b_2}{z + a_3 -} \cdots$$

and if the odd part exists go to the form

$$\frac{g_0}{z+} \; \frac{f_1}{1+} \; \frac{g_1}{z+} \; \frac{f_2}{1+} \; \frac{g_2}{z+} \cdots.$$

Stiltjies often used the transformed expression,

$$\frac{a_0}{\alpha_1 z+} \; \frac{1}{\alpha_2 +} \; \frac{1}{\alpha_3 z+} \; \frac{1}{\alpha_4 +} \cdots$$

with linkage to the Stiltjies moment problem which yields a unique solution provided $\alpha_s > 0$ and in addition $\sum \alpha_s = \infty$.

Omitting details we find for the mean, with $N = 1, 2, \cdots$

$$E(\hat{P}_N) = \frac{N}{N+} \; \frac{NQ/P}{1+} \; \frac{1/P}{N+} \; \frac{(N+1)Q/P}{1+} \; \frac{2/P}{N+} \; \frac{(N+2)Q/P}{1+} \cdots \qquad (6)$$

which converges to $E(\hat{P}_N)$, the odd convergents forming a monotonic decreasing system of bounds, the even convergents forming a monotonic increasing system of bounds. Note that for $s \geq 1$, $f_s = (N + s - 1)Q/P$, $g_s = s/P$.

Example 1. $n = 10$, $P = Q = 1/2$.

s	χ_s	ω_s	Approx
1	1	1	1.0
2	1	2	0.5
3	12	22	0.5455
4	23	44	0.5227
5	278	528	0.5265
6	554	1056	0.5246
7	7208	13728	0.5251

(Correct value is 0.524877).

Example 2.

From (6) with $P = Q = 1/2$, $n = 1$, we have the interesting result

$$\ln 2 = \frac{1}{1+}\ \frac{1}{1+}\ \frac{2}{1+}\ \frac{2}{1+}\ \frac{4}{1+}\ \frac{3}{1+}\ \frac{6}{1+}\ \frac{4}{1+}\ \cdots$$

The 7th and 8th convergent give

$$\frac{262}{384} < \ln 2 < \frac{134}{192}.$$

Comments: Table 1 is an excerpt from a larger study in which $P = 0.1(0.1)0.9$, $N = 1(1)5(5)25$.

Table 1 Moments and moment ratios of the maximum likelihood estimator $\hat{P}_N$ of P for the geometric distribution

P	N	μ_1'	σ	$\sqrt{\beta_1}$	β_2
0.2	1	0.40235948	0.32681111	0.96230881	2.47111638
	2	0.29882026	0.21063542	1.67336417	5.76548538
	3	0.26294240	0.15415406	1.79743388	7.33334688
	4	0.24568587	0.12333648	1.72046403	7.53208684
	5	0.23572317	0.10433004	1.59713176	7.19127205
	10	0.21695652	0.06496800	1.13691441	5.33096651
	15	0.21109334	0.05068584	0.91266049	4.51833760
	20	0.20824021	0.04289374	0.78156998	4.11551117
	25	0.20655377	0.03783510	0.69387369	3.87934404
0.5	1	0.69314718	0.31904155	-0.23010833	1.34525868
	2	0.61370564	0.25882814	0.33079339	1.87684585
	3	0.57944154	0.21662248	0.52940992	2.45587271
	4	0.56074461	0.18806299	0.60293137	2.82600334
	5	0.54906924	0.16769163	0.62520331	3.04318901
	10	0.52487740	0.11631758	0.56789458	3.30269177
	15	0.51662995	0.09394836	0.49545323	3.28006679
	20	0.51248445	0.08085016	0.44180946	3.24001809
	25	0.50999203	0.07202105	0.40163483	3.20628352
0.8	1	0.89257421	0.21754629	-1.60591183	3.80132218
	2	0.85940636	0.19587335	-0.88762215	2.29375668
	3	0.84356185	0.17478478	-0.58709260	2.09482289
	4	0.83433679	0.15830117	-0.42446722	2.12274659
	5	0.82831606	0.14539716	-0.32465265	2.19731975
	10	0.81505405	0.10818181	-0.13115389	2.50579685
	15	0.81024349	0.08974115	-0.07444960	2.65651315
	20	0.80776132	0.07831247	-0.04932740	2.73901169
	25	0.80624702	0.07035781	-0.03570286	2.79020452

3.3.2 Tabulations of the Moments of the Geometric Distribution's Maximum Likelihood Estimator $\hat{P}_N$ of P

Table 1 gives a brief account of the low order moments of $\hat{P}_N$ including the moment ratios $\sqrt{\beta_1}$ = skewness = $\mu_3/\mu_2^{3/2}$ and β_2 = kurtosis = μ_4/μ_2^2.

The entries have all been evaluated by the series approach, in some cases some 200 terms were required to meet the accuracy of one digit in the seventh decimal place. The mean was checked using the continued fraction in (6), for which only 50 convergents were needed. For Q (> 0) small and a small sample, there is a large negative skewness and large kurtosis; these extremes are moderated when the sample size increases to around 50. For Q near to unity but less than one, both the skewness and the kurtosis reach rather extreme values, these only moderated slightly as N approaches 50. For $P = Q = 1/2$, both skewness and kurtosis increase as N increases to 10, thereafter decreasing to zero for skewness and 3 for kurtosis. Note that for the geometric distribution Q small means the mean is small, whereas Q near to unity means that the mean of the random variate is large.

3.3.3 The Variance of $\hat{P}$ and a Second Order Continued Fraction

We use the integral in (4) with $r = 2$, this being of the form

$$F_2(z) = \int_0^\infty \frac{d\sigma(t)}{(z+t)^2}$$

and from Shenton (1956, p181; equations 48 & 49, and Table on p189) we may set up convergent sequences j_s^*/j_s. The basic entities are:

$$\alpha_s = b_s(2N + b_{s+1} + b_s + b_{s-1})$$
$$\beta_s = b_s b_{s-2}\alpha_{s-1}, \qquad \delta_s = b_s b_{s-1} b_{s-2} b_{s-3} b_{s-6};$$

and $j_0^* = 1$, $j_1^* = j_2^* = b_1$, $j_s^* < 0$, $s < 0$, $j_0 = 1$, $j_1 = N^2$, $j_2 = N^2 + b_2(2N + b_2 + b_3)$.

The partial numerators of the basic continued fraction are

$$b_{2s} = (N + s - 1)Q/P, \qquad b_{2s+1} = s/P \qquad (s = 1, 2, \cdots)$$

$$b_1 = 1.$$

The convergents, j_s^* and j_s follow the recurrence scheme.

$$\omega_{2s-1} = z_1 z_2 \omega_{2s-2} + \alpha_{2s-1}\omega_{2s-3} - \beta_{2s-1}\omega_{2s-5} - z_1 z_2 \gamma_{2s-1}\omega_{2s-6} + \delta_{2s-1}\omega_{2s-7},$$
$$\omega_{2s} = \omega_{2s-1} + \alpha_{2s}\omega_{2s-2} - \beta_{2s}\omega_{2s-4} - \gamma_{2s}\omega_{2s-5} + \delta_{2s}\omega_{2s-6}. \qquad (z_1 = z_2 = N)$$

Note that Shenton (1956) proves that the odd convergents $j_{2s+1}^*/j_{2s=1}$ form a monotonic decreasing sequence of bounds, whereas j_{2s}^*/j_{2s} form monotonic increasing bounds.

Table 2 gives a set of results for $P = 0.1(0.2)0.9$ and $N = 1, 5, 10, 25, 50$.

Table 2. Upper and Lower Bound for $E(\hat{P}^2)$,
$\sigma(\hat{P})$, using 2nd order continued fraction

P	n	$E(\hat{P}^2)$ U	$E(\hat{P}^2)$ L	$\sigma(\hat{P}^2)$	σ by series
	1	0.923560	0.923560	0.156180	0.156180
	5	0.850276	0.850276	0.110616	0.110616
0.9	10	0.832082	0.832082	0.083870	0.083870
	25	0.819352	0.819352	0.055341	0.055341
	50	0.814767	0.814767	0.039688	0.039688
	1	0.760969	0.760969	0.261425	0.261425
	5	0.571927	0.571927	0.163420	0.163420
0.7	10	0.532752	0.532752	0.119088	0.119088
	25	0.507453	0.507453	0.076272	0.076272
	50	0.498776	0.498776	0.054095	0.054095
	1	0.582251	0.582126	0.319059	0.319042
	5	0.329598	0.329597	0.167692	0.167692
0.5	10	0.289026	0.289026	0.116318	0.116318
	25	0.265279	0.265279	0.072021	0.072021
	50	0.257572	0.257572	0.050482	0.050482
	1	0.382439	0.373580	0.340824	0.338995
	5	0.137202	0.137186	0.135481	0.135450
0.3	10	0.111286	0.111286	0.088024	0.088023
	25	0.097938	0.097938	0.052412	0.052412
	50	0.093874	0.093874	0.036279	0.036279
	1	0.179422	0.088586	0.334722	0.280993
	5	0.018553	0.018320	0.062188	0.060677
0.1	10	0.013329	0.013327	0.035798	0.035775
	25	0.011171	0.011171	0.020330	0.031976
	50	0.010562	0.010562	0.013884	0.013884

Comments: Thirty terms of the continued fraction were used. Agreement with $\sigma(\hat{P})$ computed by series is excellent for Q small but deteriorate for Q near to unity, especially for small sample sizes.

4 Cumulants for the Geometric Distribution

4.1 A Recursion

We have for the cumulants

$$K(\alpha) = \exp\left\{\frac{\alpha^2}{2!}K_2 + \frac{\alpha^3}{3!}K_3 \cdots\right\} = \frac{Pe^{-\alpha Q/P}}{1 - Qe^{\alpha}}$$

so that

$$(1 - Qe^{\alpha})e^{K(\alpha)} = Pe^{-\alpha Q/P}.$$

Differentiating with respect to α, we have

$$(1 - Qe^{\alpha})\frac{\partial K(\alpha)}{\partial \alpha} = -\frac{Q}{P}(1 - e^{\alpha})$$

and equating coefficient of $\alpha^r/r!$ we have the recurrence

$$K_{r+1} = Q\{K_{r+1} + \binom{r}{1}K_r + \binom{r}{2}K_{r-1} + \cdots + \binom{r}{r-1}K_2 + \frac{1}{P}\} \tag{7}$$

where $r = 1, 2, \cdots$, with $K_2 = Q/P^2$. We find that

$$K_r = h_{r-1}(Q)/P^r.$$

Tabulation of $h_{r-1}(Q)$ is given in Table 3.

Table 3. Cumulant Components for the Geometric Random Variable

K_s	Q	Q^2	Q^3	Q^4	Q^5	Q^6	Q^7	Q^8	Q^9	Q^{10}	Q^{11}
2	1										
3	1	1									
4	1	4	1								
5	1	11	11	1							
6	1	26	66	26	1						
7	1	57	302	302	57	1					
8	1	120	1191	2416	1191	120	1				
9	1	247	4293	15619	15619	4293	247	1			
10	1	502	14608	88234	156190	88234	146008	502	1		
11	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1	
12	1	2036	152637	2203488	9738114	15724248	9738114	2203488	152637	2036	1

It will be seen that the elements of h_{r-1} are positive integers and that they are symmetric. For example,

$$h_2(Q) = Q + Q^2,$$
$$h_3(Q) = Q + 4Q^2 + Q^3.$$

If we consider $\lim_{P\to 0} P^{r+1}K_{r+1}(Q)$ it follows that

$$h_r(1) = r!. \qquad (r = 1, 2, \cdots)$$

Defining

$$h_r(Q) = \sum_{s=1}^{r} h_s^{(r)} Q^s$$

we see that the components $h_1^{(r)}$, $h_2^{(r)}$, $\cdots$, $h_r^{(r)}$ represent a partition of r parts from $r!$.

The components $h_s^{(r)}$ are Eulerian numbers. We encounter them in our book on maximum likelihood estimators (Shenton and Bowman, 1977) in connection with Fisher's logarithmic distribution which has the probability function

$$Pr(X = x)\alpha\theta^x/x \qquad (x = 1, 2, \cdots)$$

and $\alpha^{-1} = \ln[1/(1-\theta)]$, $0 \le \theta < 1$. For the central moments

$$\mu_s = \frac{1}{T_0}\sum_{r=0}^{s}(-1)^r \mu^r T_{s-r}(\theta),$$

where

$$T_0(\theta) = \alpha^{-1}, \quad \mu = E(\bar{x}) = -\frac{\theta}{(1-\theta)\ln(1-\theta)}.$$

It is shown (Shenton and Bowman, 1977, chapter 5) that in terms of Eulerian numbers,

$$(1-\theta)^3 T_3(\theta) = \theta + \theta^2, \quad (1-\theta)^4 T_4(\theta) = \theta + 4\theta^2 + \theta^3,$$

and in general

$$(1-\theta)^s T_s(\theta) = h_{s-1}(\theta).$$

Eulerian numbers are studied in Riordan (1958, p215), Comtet (1974, p243). A tabulation is given in David, Kendall and Barton (1966). For references associated with "Power Series" distributions, see Patil (1986), and Douglas (1980).

We shall give some properties of Eulerian numbers in the sequel.

4.2 An Approximate Solution for the rth Cumulant K_r Associated with the Geometric Distribution

The binomial coefficients in the recurrence (7) suggest a possible solution of the form

$$K_r^* = [\lambda(Q)]^r (r-1)!. \quad (r = 2, 3, \cdots)$$

Substitution leads to

$$\lambda^{r+1} r! = Q\left\{\lambda^{r+1} r! + \frac{\lambda^r r!}{1!} + \frac{r!\lambda^{r-1}}{2!} + \frac{r!\lambda^{r-2}}{3!} + \cdots + \frac{r!\lambda^2}{(r-1)!} + \frac{1}{P}\right\}$$

so that

$$1 = Q\left\{1 + \frac{1}{\lambda 1!} + \frac{1}{\lambda^2 2!} + \frac{1}{\lambda^3 3!} + \cdots + \frac{1}{\lambda^{s-1}(s-1)!} + \frac{1}{\lambda^s s! P}\right\},$$

suggesting for large r the solution

$$\lambda(Q) = 1/(\ln \frac{1}{Q}),$$

and the approximation

$$K_r^* = [\lambda(Q)]^r (r-1)!.$$

Table 4. Approximants K_r^* to K_r and its Errors.

	$Q = 0.2$		$Q = 0.5$		$Q = 0.8$	
s	K_s	$K_s - K_s^*$	K_s	$K_s - K_s^*$	K_s	$K_s - K_s^*$
2	0.3125	0.7356	2	0.8137e-1	20	0.8313e-1
3	0.4688	0.1099e-1	6	0.5561e-2	180	0.1852e-2
4	0.8984	0.4197e-2	26	0.7419e-2	2420	0.8235e-2
5	2.2266	0.4071e-2	150	0.2529e-2	43380	0.8778e-3
6	6.9043	0.2611e-3	1082	0.3037e-2	972020	0.3865e-2
7	25.7373	0.2954e-2	9366	0.2495e-2	26136180	0.9158e-3
8	111.9519	0.1349e-2	94586	0.2537e-2	819890420	0.3980e-2
9	556.4868	0.3080e-2	1091670	0.4185e-2	29394187380	0.1652e-2
10	3111.8695	0.4436e-2	14174522	0.3246e-2	1185549324020	0.7059e-2
11	19335.1016	0.3181e-2	204495126	0.1054e-1	53129445912180	0.4554e-2
12	132149.3226	0.1428e-1	3245265146	0.4900e-2	26190490452984 20	0.1906e-1

Table 4 illustrates a surprising accuracy of K_r^* for $1 > Q \geq 1/2$. Note that the approximation K_r^* satisfies the difference-differential equation for K_r.

4.3 The Components of $h_r(Q)$

In the difference-differential equation for K_r set $Q = e^\alpha$. Then

$$K_{r+1}(e^\alpha) = \frac{\partial K_r(e^\alpha)}{\partial \alpha} = \left(\frac{\partial}{\partial Q}\right)^{r-1} \frac{e^\alpha}{(1-e^\alpha)^2}$$

or

$$K_{r+1}(e^\alpha) = \left(\frac{\partial}{\partial \alpha}\right)^r \frac{1}{1-e^\alpha}.$$

Hence

$$h_r(e^\alpha) = (1-e^\alpha)^{r+1}\left(\frac{\partial}{\partial \alpha}\right)^r \frac{1}{1-e^\alpha}. \tag{8}$$

This type of equation has a long history, going back three centuries ago, when Euler, according to an account in Hardy (1949), gives the equation

$$1^m e^{-y} - 2^m e^{-2y} + 3^m e^{-3y} - \cdots = (-1)^m \left(\frac{d}{dy}\right)^m \frac{1}{e^y+1}.$$

Equating powers of e^α in (8) we find that with $\alpha < 0$

$$\begin{aligned} h_r(x) &= h_1^{(r)}x + h_2^{(r)}x^2 + \cdots + h_r^{(r)}x^r, \\ h_1^{(r)} &= 1^r, \\ h_2^{(r)} &= 2^r - \binom{r+1}{1} 1^r, \\ h_3^{(r)} &= 3^r - \binom{r+1}{1} 2^r + \binom{r+1}{2} 1^r, \end{aligned}$$

and in general

$$h_s^{(r)} = \sum_{u=0}^{s-1} (-1)^u \binom{r+1}{u} (s-u)^r \quad .(s = 1, 2, \cdots, r) \tag{9}$$

When r is odd, say $r = 2R-1$, the entries in Table 3 suggest a maximum value for $h_s^{(2R-1)}$ when $s = R$, the polynomial being symmetric. Similarly when r is even, say $2R$, $h_s^{(2R)}$ reaches maximums at $s = R$ and $s = R+1$, the polynomials also being symmetric. These properties are not obvious to follow from (9).

Consider the first case, ie. $r = 2R-1$. Then for $e^{-R\alpha}h_{2R-1}$ we have

$$J_R(\alpha) = e^{-R\alpha}(1-e^\alpha)^{2R} \left(\frac{\partial}{\partial\alpha}\right) \frac{1}{1-e^\alpha}$$

should be an even function. But

$$J_R(\alpha) = (e^{-\alpha/2} - e^{\alpha/2})^{2R} \left(\frac{\partial}{\partial\alpha}\right)^{2R-1} \frac{1}{1-e^\alpha}$$

and

$$J_R(-\alpha) = -(e^{-\alpha/2} - e^{\alpha})^{2R} \left(\frac{\partial}{\partial\alpha}\right)^{2R-1} \frac{e^\alpha}{e^\alpha - 1} = (e^{-\alpha/2} - e^{\alpha})^{2R} \left(\frac{\partial}{\partial\alpha}\right)^{2R-1} \frac{1}{1-e^\alpha} = J_R(\alpha).$$

A similar proof applies to $e^{-R\alpha}h_{2R}$. That the components $h_s^{(r)}$ increase as positive integers up to the maximum may be demonstrated as follows. We have for the difference-differential equation for K_r

$$h_r = Q(1-Q)\frac{\partial}{\partial Q}h_{r-1} + rQh_{r-1}, \qquad (h_r = h_r(Q); r = 2, 3, \cdots)$$

so that for the coefficient of Q^s,

$$h_s^{(r)} = s h_s^{(r-1)} + (r-s+1) h_{s-1}^{(r-1)}. \quad (r = 2, 3, \cdots; s = 2, 3, \cdots.)$$

Hence

$$h_s^{(r)} = s(h_s^{(r-1)} - h_{s-1}^{(r-1)}) + (r+1) h_{s-1}^{(r-1)},$$

and

$$\begin{aligned} h_s^{(r)} - h_{s-1}^{(r)} &= s(h_s^{(r-1)} - h_{s-1}^{(r-1)}) - (s-1)(h_{s-1}^{(r-1)} - h_{s-2}^{(r-1)}) + (r+1)h_{s-1}^{(r-1)} - (r+1)h_{s-2}^{(r-1)} \\ &= s(h_s^{(r-1)} - 2h_{s-1}^{(r-1)} + h_{s-2}^{(r-1)}) + (h_{s-1}^{(r-1)} - h_{s-2}^{(r-1)}) + (r+1)(h_s^{(r-1)} - h_{s-1}^{(r-1)}). \end{aligned}$$

Now there is evidence in Table 3 that the first and second differences of $h_s^{(r)}$ are positive on either side of the mean (or maximum value). Hence induction now proves this for the general case.

4.4 Approximants to the Logarithmic Function

Let

$$L(P) = -[\ln(1-P)]/P \qquad (0 < P < 1).$$

Then cumulants associated with the geometric distribution, lead to the approximants

$$L_r(P) = [(r-1)!/h_r(P)]^{1/r} = \left[\frac{(r-1)!}{\sum_{s=1}^{r} Q^s h_s^{(r)}}\right]^{1/r} \qquad (r = 1, 2, \cdots; 0 < Q < 1, Q + P = 1)$$

and $h_s^{(r)}$ is defined in (9).

It is more informative to express the denominator in powers of the parameter P. For example,

$$(1) \qquad L_3(P) = \left(1 - \frac{3P}{2} + \frac{P^2}{2}\right)^{-1/3} = \sum_{s=0}^{\infty} l_s^{(3)} P^s$$

and

$$6(r+1) l_{r+1}^{(3)} = 3(3r+1) l_r^{(3)} - (3r-1) l_{r-1}^{(3)} \qquad (l_0^{(3)} = 1)$$

$$l_2^{(3)} = 1/3, \qquad l_3^{(3)} = 1/4, \qquad l_4^{(3)} = 29/144.$$

$$(2) \qquad L_4(P) = \left(1 - 2P + \frac{7P^2}{6} - \frac{P^3}{6}\right)^{-1/4} = \sum_{s=0}^{\infty} l_s^{(4)} P^s$$

$$(r+1) l_{r+1}^{(4)} = \left(2r + \frac{1}{2}\right) l_r^{(4)} - \frac{7}{12}(2r-1) l_{r-1}^{(4)} + \frac{1}{24}(4r-5) l_{r-2}^{(4)}$$

$$l_1^{(4)} = 1/2, \qquad l_2^{(4)} = 1/3, \qquad l_3^{(4)} = 1/4, \qquad l_4^{(4)} = 1/5, \qquad l_5^{(4)} = 115/576.$$

$$(3) \quad L_6(P) = \left(1 - 3P + \frac{13P^2}{4} - \frac{3P^3}{2} + \frac{31P^4}{120} - \frac{P^5}{120}\right)^{-1/6} = \sum_{s=0}^{\infty} l_s^{(6)} P^s.$$

$$(r+1)l_{r+1}^{(6)} = \frac{1}{2}(6r+1)l_r^{(6)} - \left(\frac{13r}{4} - \frac{13}{6}\right) l_{r-1}^{(6)} + \left(\frac{3r}{2} - \frac{27}{12}\right) l_{r-2}^{(6)} - \left(\frac{31r}{120} - \frac{217}{360}\right) l_{r-3}^{(6)} + \left(\frac{r}{120} - \frac{19}{720}\right) l_{r-4}^{(6)}$$

$$l_s^{(6)} = 1/(s+1), \quad s = 0, 1, \cdots, 5; \quad l_6^{(6)} = 3703/25920.$$

From an extended study of $L_r(P)$ for $r = 2$ to 12, we have found that $L_{2r}(P)$ and $L_{2r+1}(P)$, term, by term, agree with the terms in $L_r(P)$ up to $P^r/(r+1)$. Table 5 gives two examples, $r = 6$ and $r = 10$. Let

$$L_r(P) = \sum_{s=0}^{t} l_s^{(r)} P^s + l_{r+1}^{(r)} P^{r+1} + l_{r+2}^{(r)} P^{r+2} + \cdots,$$

with $l_0^{(r)} = 1$ and t a known positive integer. Thus the first $t+1$ terms agree with these of $L(P)$. Table 5 gives an example showing that the discrepancies between unity and $l_{t+2}^{(r)}(t+2)$, and $l_{t+3}^{(r)}(t+3)$ up to a dozen or so terms are remarkably small. Table 6 gives some idea of the accuracy of the logarithmic approximation when $P = Q = 1/2$.

Table 5 Coefficients $l_s^{(r)}$, $r = 6, 10$

s	$l_s^{(r)}$	App	$l_s^{(r)}$/App
	$r = 6$		
1	1/2	1/2	1
2	1/3	1/3	1
3	1/4	1/4	1
4	1/5	1/5	1
5	1/6	1/6	1
6	3703/25920	25921/25920	1.000038580
7	6481/51840	6481/6480	1.000154321
8	345721/3110400	345721/345600	1.000350116
9	69161/691200	69161/69120	1.000593171
10	565997/6220800	6225967/6220800	1.000830601
11	518921/6220800	518921/518400	1.001005015
12	517355267/6718464000	6725618471/6718464000	1.001064897
13	960711311/13436928000	6724979177/6718464000	1.000969742
14	5378488001/80621568000	5378488001/5374771200	1.000691527
15	671990191/10749542400	671990191/671846400	1.000214024
16	379217709029/6449725440000	6446701053493/6449725440000	0.9995310829
17	238554941351/4299816960000	238554941351/238878720000	0.9986445898
18	27429174965939/522427760640000	521154324352841/522427760640000	0.9975624644
19	104098631581621/2089711042560000	104098631581621/104485552128000	0.9962968991
20	1146734289121949/25076532510720000	8027140023853643/8358844170240000	0.9603169841
	$r = 10$		
1	1/2	1/2	1
2	1/3	1/3	1
3	1/4	1/4	1
4	1/5	1/5	1
5	1/6	1/6	1
6	1/7	1/7	1
7	1/8	1/8	1
8	1/9	1/9	1
9	1/10	1/10	1
10	3958691/43545600	43545601/43545600	1.000000023
11	7257601/87091200	7257601/7257600	1.000000138
12	703429229/9144576000	9144579977/9144576000	1.000000435
13	1306369247/18289152000	1306369247/1306368000	1.000000955
14	2438557511/36578304000	2438557511/2438553600	1.000001604
15	914459519/14631321600	914459519/914457600	1.000002099

Table 6 Approximations to $-\ln(1-P)$ when $P = Q = 1/2$

r	App=$\{(r-1)!/K_r\}^{1/r}$	ln(2)-App
2	0.70710678118654752440	0.01395960062660221498
3	0.69336127435063470487	0.00021409379068939545
4	0.69309772861787781251	-0.00004945194206749691
5	0.69314484315514639287	-0.00000233740479891655
6	0.69314750482185122325	0.00000032426190591383
7	0.69314720693303394244	0.00000002637308863302
8	0.69314717823564334670	-0.00000000232430196272
9	0.69314718026469516584	-0.00000000029525014358
10	0.69314718057581756174	0.00000000001587225232
11	0.69314718056319376018	0.00000000000324845076
12	0.69314718055985809640	-0.00000000000008721302
13	0.69314718055991033311	-0.00000000000003497631
	ln(2) = 0.69314718055994530942	

5 The Central Moments of the Cumulant Components

We have

$$\frac{1}{r!}\sum_{s=0}^{\infty}\left(h_s^{(r)}-\frac{r+1}{2}\right)^m=\mu_m(\underline{h})$$

so that the generating function is

$$G_r(\alpha,h)=\frac{1}{r!}e^{-(r+1)\alpha/2}(1-e^{\alpha})^{r+1}\left(\frac{d}{d\alpha}\right)^r\frac{1}{1-e^{\alpha}}$$

and setting $\alpha/2=\beta$, this becomes

$$\frac{1}{r!}(\sinh\beta)^{r+1}\left(-\frac{d}{d\beta}\right)^r\coth\beta \qquad (r=1,2,\cdots)$$

in terms of hyperbolic function. Note that it may be useful to recall that

$$\coth\beta=\frac{1}{\beta}+\frac{\beta}{3}-\frac{\beta^3}{45}+\frac{2\beta^5}{945}+\cdots+\frac{2^{2m}B_{2m}\beta^{2m-1}}{(2m)!} \qquad (|\beta|<\pi)$$

in terms of Bernoulli numbers, with

$$B_0=1,\ \ B_1=-\frac{1}{2},\ \ B_2=\frac{1}{6},\ \ B_4=-\frac{1}{30}.$$

For examples of the moments we have

$$\mu_1'(h_{s-1})=s/2,$$

$$
\begin{aligned}
\mu_2(h_{s-1}) &= s/12, \\
\mu_3(h_{s-1}) &= 0, \\
\mu_4(h_{s-1}) &= s(5s-2)/240, \qquad (s > 4) \\
\mu_5(h_{s-1}) &= 0, \\
\mu_6(h_{s-1}) &= \frac{5s^3}{(4!)^2} - \frac{s^2}{96} + \frac{s}{252}, \qquad (s > 6) \\
\beta_2 &= 3 - \frac{6}{5s}.
\end{aligned}
$$

If $s = 50$, then $\beta_2 = 2.976$ approximately. A set of values are given in Table 7. Considering $\sqrt{\beta_1}$ and β_2 it appears that the distribution is near to the normal.

Table 7. The Central Moments Associated with K_s

s	Mean	Variance	μ_4		μ_6		β_2
3	3/2	1/4					
4	2	1/3					
5	$2\frac{1}{2}$	5/12	23/48	0.48			2.76
6	3	1/2	7/10	0.70			2.80
7	$3\frac{1}{2}$	7/12	77/80	0.96	479/192	2.49	2.83
8	4	2/3	19/15	1.27	80/21	3.81	2.85
9	$4\frac{1}{2}$	3/4	129/80	1.61	2473/448	5.52	2.87
10	5	5/6	2	2.00	215/28	7.68	2.88
11	$5\frac{1}{2}$	11/12	583/240	2.43	4631/448	10.34	2.89
12	6	1	29/10	2.90	569/42	13.55	2.90
13	$6\frac{1}{2}$	13/12	273/80	3.41	23335/1344	17.36	2.91
14	7	7/6	119/30	3.97	131/6	21.83	2.91
15	$7\frac{1}{2}$	5/4	73/16	4.56	36305/1344	27.01	2.92

6 Conclusion

The cumulants may be set up using $K_{r+1} = Q\frac{\partial K_r}{\partial Q}$ with $K_1 = Q/P$or by using the recurrence given in §4.2; the latter is more complicated.

The cumulant components $h_s^{(r)}$ may be set up from using

$$h_s^{(r)} = sh_s^{(r-1)} + (r-s+1)h_{s-1}^{(r-1)}, \qquad (r = 2, 3, \cdots)$$

with $h_1 = Q$, $h_1^{(1)} = 1$, $h_2^{(1)} = 0$.

There is the approximate formula,

$$\frac{\ln(Q^{-1})}{1-Q} \sim \left\{ \frac{(r-1)!}{\sum_{s=1}^{r-1} h_s^{(r-1)} Q^s} \right\}^{1/r}, \qquad (r = 2, 3, \cdots; 0 < Q < 1)$$

where $h_s^{(r-1)} = \sum_{u=0}^{s-1}(-1)^u \binom{r+1}{u} (s-u)^r$.

Notice the difference-differential equations,

$$y_r(Q) = Q\sum_{s=1}^{r}\binom{r}{s} P^{s-1} y_{r-s}(Q) \qquad (h_0 = 1; r = 2, 3, \cdots)$$

$$\frac{\partial y_r}{\partial Q} - \frac{y_r}{Q} + \sum_{s=1}^{r}\binom{r}{s} P^{s-2}\left\{\frac{P\partial y_{r-s}}{\partial Q} - s y_{r-s}\right\} \qquad (r = 2, 3, \cdots)$$

with solution $y_r = h_r = h_r(Q)$.

With respect to the "positive" binomial, probability generating function being $(pt+q)^n$, its cumulants $\bar{K}_r$ are given by

$$\bar{K}_r = -nq^{r+1}h_r(-p/q), \qquad (r = 1, 2, \cdots)$$

using $P = 1/q$, $Q = -p/q$. In this case the approximate formula $K_r^* = \lambda^*(r-1)!$ is not valid, but $\bar{K}_r$ does involve Eulerian numbers.

There would be interest in setting up an asymptotic expression for the error $L(P) - L_r(P)$, $r \to \infty$. Another possibility less in the direction of the statistical properties of partitions of the factorial.

Acknowledgment

The second authors research was sponsored by the Applied Mathematical Sciences Research Program, Oak Ridge National Laboratory, is operated by UT-Battelle, LLC for the U. S. Department of Energy, under contract number DE-AC05-00OR22725.

Reference

Aitken, A.C. and Gonin, H.T. (1934), On Fourfold Sampling with and without Replacement, *Proc. Royal Society of Edinbugh*, LV-part II, (No.11), 114-125.

Bowman, K.O. and Shenton, L.R. (1989), *Continued Fractions in Statistical Applications*, Marcel Dekker, Inc.

Bowman, K.O. and Shenton, L. R., (1998). Asymptotic Skewness and the Distribution of Maximum Likelihood Estimators, *Commun. Statist. Theory and Method*, 27(11), 2743-2760.

Bowman, K.O. and Shenton, L. R., (1999). The Asymptotic Kurtosis for Maximum Likelihood Estimators, *Commun. Statist. Theory and Method*, 28(11), 2641-2654.

Bowman, K.O. and Shenton, L.R., (2000). Maximum Likelihood and The Weibull Distribution, *Far East J. Theo. Stat.*, 4(2), 391-422.

Bowman, K.O. and Shenton, L.R., (2001). Weibull Distributions when the Shape Parameter is Defined, *CSDA*, (to appear).

Bowman, K.O. and Williams, P.T., (2000). *Technical Basis for Statistical Models of Extended* K_{Ic} *and* K_{Ia} *Fracture Toughness Database for RPV Steels*, ORNL/NRC/LTR-99/27.

Brezinski, C., (1978). *A Bibliography on Padé Approximation and Related Subjects, Addendum 1,* du Laboratoire de Calcul de l'Université des Sciences et Techniques de Lille.

Brezinski, C., (1980a). *A Bibliography on Padé Approximation and Related Subjects, Addendum 2,* du Laboratoire de Calcul de l'Université des Sciences et Techniques de Lille.

Brezinski, C., (1980b). *Padé Type Approximation and General Orthogonal Polynomials*, Birkhäuser Verlag.

Comtet, L. (1974), *Advanced Combnatorics*, D. Reidel, Boston.

David, F.N., Kendall, M.G. and Barton, D.E. (1966, *Symmetric Function and Allied Tables*, Cambridge University Press, Cambridge.

Douglas, J.B. (1980), *Statistical Distributions, Analysis with Standard Contagious Distributions*, Vol 4, International Co-operative publishing House, Maryland.

Hardy, G.H. (1949), *Divergent Series*, Oxford, Clarendon Press.

Johnson, N.L., Kotz, S. and Kemp, A.W. (1993). *Univariate Discrete Distributions, Second Edition*, John Wiley & Sons, New York.

Patil, G.P. (1986), Power Series, Encyclopedia of Statistical Sciences, vol 7, p130.

Riodan, J. (1958), *An Introduction to Combinatorial Analysis*, John Wiley & Sons, New York.

Shenton, L.R. (1956), A Determinantal Expansion for a Class of Definite Integrals. Part 5: Recurrence Relations. *Proc. Edinb. Math. Soc.*, 9, 170-182.

Shenton, L.R. (1958), Generalized Algebraic Continued Fractions related to definite Integrals, *Proc. Edinb. Math. Soc.*, 9, 170-182.

Shenton, L.R. and Bowman, K.O. (1971), Continued Fractions for the Psi Function and Its Derivatives, *SIAM Jour. Appl. Math.*, 20, 547-554.

Shenton, L.R. and Bowman, K.O. (1977), *Maximum Likelihood Estimation in Small Samples*, England: Charles Griffin and Co., Ltd.

Shohat, J.A. and Tamarkin, J.D. (1943), *Problem of Moments*, American Mathematical Society.

Wall, H.S. (1948), *Analytic Theory of Continued Fractions*, Chelsea Publishing Company, Bronx, New York.