THE DERIVATIVE OF A CONTINUED FRACTION

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Abstract

The paper considers second order continued fractions associated with (I) the Psi function $\psi(z)$, (II) the continued fraction component in $\ln \Gamma(z)$ due to Stieltjes. The second order sequences $k_s^*/k_s$ provide approximants, some of which are remarkably close. In addition a series form for the convergent $\chi_s/\omega_s$ associated with a continued fraction provides an expression for the derivatives of a continued fraction. The implementation uses a Maple code for derivatives.

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1 Introduction

We have discussed this subject in our paper Shenton and Bowman (2005, p.21). For a continued fraction it is assumed that

\[ \frac{b_1}{z+1} + \frac{b_2}{z+1} + \frac{b_3}{z+1} + \cdots = \int_0^\infty \frac{d\sigma(u)}{u+z} \quad (\Re(z) > 0) \]

the integral of Stieltjes form, the partial numerators being real and positive. The notation for the continued fraction is

\[ \frac{b_1}{z} + \frac{b_2}{z} + \frac{b_3}{z} + \cdots \]

In general the partial numerators are positive reals.

In this paper we introduce new examples of second order continued fractions. We are given a series

\[ \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_3}{z^4} - \frac{c_4}{z^6} + \cdots \]

which in some domain may be convergent or divergent. A good reference is given in Borel (1928); see also Bowman and Shenton (1989).

To highlight the basic procedures we give a simple example. For the Psi function \( \psi(\cdot) \) we have the continued fraction expression

\[ \psi(z) = \ln z - \frac{1}{2z} - \frac{a_0}{z^2 + 1} - \frac{a_1}{z^2 + 1} - \frac{a_2}{z^2 + 1} - \cdots \quad (\Re(z) > 0) \]

where

\[ a_0 = \frac{1}{12}, \quad a_1 = \frac{1}{10}, \quad a_2 = \frac{79}{210}, \quad a_3 = \frac{1205}{1659}, \quad a_4 = \frac{262445}{209429}, \quad a_5 = \frac{2643428417511}{1429053441530}. \]

There are other interesting forms of (1).

(a) Integral form

\[ \psi(z) = \ln z - \frac{1}{2z} - \int_0^\infty \frac{du}{(z^2 + u)(e^{2\pi\sqrt{u}} - 1)} \]

given by Shenton and Bowman (1971, p.552). The integral is Stieltjes and subscribes to the form

\[ \int_0^\infty \frac{d\sigma(u)}{u + z^2} \]
which arises for a class of continued fraction (Wall, 1948). It suggests a continued fraction

\[
\frac{d_1}{z^2+1} + \frac{d_2}{z^2+1} + \frac{d_3}{z^2+1} + \cdots
\]

(b) Asymptotic series

\[
\psi(z) = \ln z - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}} \quad (z \to \infty \text{ in } [\arg] < \pi)
\]

and in more detail

\[
\psi(z) \sim \ln z - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{20z^4} - \frac{1}{252z^6} + \frac{1}{210z^8} - \frac{1}{12z^{10}} \cdots,
\]

this series, extended, may be used to set up the continued fraction using Wall’s algorithm (Wall, 1948, p.194).

We now return to expression (1) for \(\psi(z)\) and differentiate with respect to \(z\), finding for the derivative of \(\psi(z)\)

\[
\psi_1(z) = \frac{1}{z} + \frac{1}{2z^2} - \frac{d}{dz} \left( \frac{a_0}{z^2+1} + \frac{a_1}{z^2} + \frac{a_2}{1+1} + \cdots \right).
\]

Thus

\[
\psi_1(z) - \frac{1}{z} - \frac{1}{2z^2} = -\frac{d}{dz} \left( \frac{a_0}{z^2+1} + \frac{a_1}{z^2} + \frac{a_2}{1+1} + \cdots \right)
\]

so that the second order continued fraction

\[
\frac{d}{d\omega} \left( \frac{a_0}{\omega+1} + \frac{a_1}{\omega+1} + \frac{a_2}{1+1} + \cdots \right)
\]

has convergent \(k_s^*(\omega)/k_s(\omega), s = 1, 2, \cdots\), to the function \(F(z)\) where

\[
F(z) = \frac{1}{2z} \left\{ \psi_1(z) - \frac{1}{z} - \frac{1}{2z^2} \right\}.
\]

We shall define \(F(z)\) to be the limiting value of the 2nd order continued function sequence \(k_s^*/k_s\). The algorithm of deriving \(k_s^*/k_s\) is given in Shenton (1957), and Shenton and Bowman (2005).

For examples, if \(z = 1\), then \(k_s^*(1)/k_s(1), s = 1, 2, \cdots\) converges to \(\frac{1}{2} \left( \frac{\pi^2}{6} - \frac{3}{2} \right) = 0.072\). Moreover \(k_0^* = 0, k_1^* = k_2^* = 1/12, k_1 = 1, k_2 = 1 + \frac{1}{10} \left( 2 + \frac{1}{10} + \frac{79}{210} \right)\). The approximants are \(k_1^*/k_1 = 0.08333 \cdots, k_2^*/k_2 = 0.067\).
Table 1 Approximants for $\psi_1(1)$, values of $k_s^*/k_s$

<table>
<thead>
<tr>
<th>$s$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>converges to</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_s^*/k_s$</td>
<td>0.08333</td>
<td>0.06679</td>
<td>0.07492</td>
<td>0.07088</td>
<td>0.07336</td>
<td>$\frac{1}{2} \left( \frac{\pi^2}{6} - \frac{3}{2} \right)$ or 0.07247</td>
</tr>
<tr>
<td>$\psi_1(1)$</td>
<td>1.66667</td>
<td>1.35889</td>
<td>1.64984</td>
<td>1.64175</td>
<td>1.64671</td>
<td>1.64493</td>
</tr>
</tbody>
</table>

In the table the odd convergents corresponding to 1, 3, 5, form a monotonic decreasing sequence; similarly the even convergents, corresponding to 2, 4 form a monotonic increasing sequence. Altogether the approximations are very satisfactory.

2 Examples

Example 1

In Shenton and Bowman (1971) we have given continued fractions for derivatives of the Psi functions $\psi_m(z)$, $m = 0, 1, 2, \ldots$, with $\psi(z) = d\ln \Gamma(z)/dz$. For $\psi_m(z)$ itself the first those partial numerators $C_1^{(m)}$, $C_2^{(m)}$, $C_3^{(m)}$ we stated explicitly. Looking at the continued fractions listed, it turns out that only one, that of $\psi_2(z)$ has all partial numerators defined. Thus the continued fraction is

$$
\frac{1}{z^2 + \frac{p_1}{1 + \frac{q_1}{z^2 + \frac{p_2}{1 + \frac{q_2}{z^2 + \cdots}}}}}
$$

where

$$
p_s = \frac{s^2(s + 1)}{4s + 2}, \quad q_s = \frac{s(s + 1)^2}{4s + 2}.
$$

The example is due to Stieltjes (1918, p.388) An obvious question is why $\psi_2(z)$ and its continued fraction?

$$
\psi_2(z) = -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^2} \left\{ \frac{1}{z^2 + \frac{p_1}{1 + \frac{q_1}{z^2 + \frac{p_2}{1 + \frac{q_2}{z^2 + \cdots}}}} \right\} \quad (\Re(z) > 0) \quad (2)
$$

(Note: There is a typo error in Shenton and Bowman (1971, p.548, (7b)); the first term on the right should be negative).

From (2),

$$
\left( \psi_2(z) + \frac{1}{z^2} + \frac{1}{z^3} \right) 2z^2 = -\frac{1}{z^2 + \frac{p_1}{1 + \frac{q_1}{z^2 + \cdots}}}
$$

differentiate with respect to $z$

$$
2 \frac{d}{dz} \left( z^2 \psi_2(z) + 1 + \frac{1}{z} \right) = 2z \frac{d}{d\omega} \left( \frac{1}{\omega + \frac{p_1}{1 + \frac{q_1}{\omega + \frac{p_2}{1 + \frac{q_2}{\omega + \cdots}}}} \right) \quad (\omega = z^2)
$$

resulting

$$
\left( 2\psi_2(z) + z\psi_3(z) - \frac{1}{z^2} \right) = \lim_{z \to 0} \frac{k_s^*(\omega)}{k_s(\omega)} \quad (\omega = z^2)
$$
and

\[ \frac{k_1^*}{k_1} = \frac{1}{\omega^2}, \quad \frac{k_2^*}{k_2} = \frac{1}{\omega^2 + p_1(2\omega + q_1 + p_1)}, \]

the first should be an upper bound and the second is a lower bound. Examples are given in Table 2.

<table>
<thead>
<tr>
<th>z</th>
<th>1/2</th>
<th>1.0000000</th>
<th>0.0625000</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>converges to</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>16.0000000</td>
<td>1.7777778</td>
<td>12.6137566</td>
<td>3.0634024</td>
<td>10.9613493</td>
<td>7.046952</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.0000000</td>
<td>0.5000000</td>
<td>0.7916666</td>
<td>0.7322530</td>
<td>0.6134259</td>
<td>0.685711</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0625000</td>
<td>0.0526316</td>
<td>0.0552326</td>
<td>0.0544314</td>
<td>0.0547382</td>
<td>0.054651</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 2

\[ \ln \Gamma(z) = \left( z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) + \frac{a_0}{z} + \frac{a_1}{z} + \frac{a_2}{z} + \cdots \quad (\Re(z) > 0) \]

where \( a_0 \) to \( a_6 \) are given in Wall (1948).

The continued fraction may be written

\[ \frac{\alpha_1 z}{z^2 + 1} + \frac{\alpha_2}{z^2 + 1} + \frac{\alpha_3}{z^2 + 1} + \cdots \quad (\Re(z) > 0), \]

so we write

\[ \frac{1}{z} \left\{ \ln \Gamma(z) - \left( z - \frac{1}{2} \right) \ln z + z - \frac{\ln(2\pi)}{2} \right\} = \frac{\alpha_1}{\omega} + \frac{\alpha_2}{\omega^2} + \frac{\alpha_3}{\omega^3} + \frac{\alpha_4}{\omega^4} + \cdots \quad (\omega = z^2) \]

and the right hand side is in the correct form to agree with the 2nd order continued fraction. Differentiate with respect to \( z \), we have

\[ \frac{d}{dz} \left\{ \ln \Gamma(z) - \left( z - \frac{1}{2} \right) \ln z + z - \frac{\ln(2\pi)}{2} \right\} = 2z \frac{d}{d\omega} \left\{ \frac{\alpha_1}{\omega} + \frac{\alpha_2}{\omega^2} + \frac{\alpha_3}{\omega^3} + \frac{\alpha_4}{\omega^4} + \cdots \right\}. \]

Allowing for a change in sign when we consider derivatives, the sequences \( k_1^*/k_s \) are approximants to

\[ -\frac{1}{2z} \frac{d}{dz} \left\{ \frac{\ln \Gamma(z)}{z} - \left( 1 - \frac{1}{2z} \right) \ln z + 1 - \frac{\ln(2\pi)}{2z^2} \right\} \]

\[ = -\frac{1}{2z} \left\{ \frac{\psi_1(z)}{z^2} - \frac{\ln \Gamma(z)}{z^2} - \frac{\ln(z)}{2z^2} - \left( \frac{1}{z} - \frac{1}{2z^2} \right) + \frac{\ln(2\pi)}{2z^2} \right\} \]

with

\[ k_1^* = 1/12 = k_2^*, \quad k_1 = \omega^2, \quad k_2 = \omega^2 + \alpha_2(2\omega + \alpha_3 + \alpha_2). \]
and when \( z = 1/2 \) becomes

\[
- \left\{ 2\psi_1\left(\frac{1}{2}\right) + 4 \ln 2 \right\} = 2\gamma
\]

Take \( z = 1 \). The 2nd order continued fraction \( (k_s^*/k_s) \) converges to the function

\[
F(1) = -\frac{1}{2} \left\{ \psi(1) - \left( 1 - \frac{1}{2} \right) - \frac{\ln(2\pi)}{2} \right\} = 0.07908
\]

<table>
<thead>
<tr>
<th>( z )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>converges to</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>1.3333333</td>
<td>0.9395973</td>
<td>1.2501182</td>
<td>1.0464910</td>
<td>1.1833200</td>
<td>2\gamma</td>
</tr>
<tr>
<td>1</td>
<td>0.0833333</td>
<td>0.0774363</td>
<td>0.0797778</td>
<td>0.0787481</td>
<td>0.0792193</td>
<td>0.07908</td>
</tr>
<tr>
<td>2</td>
<td>0.0052083</td>
<td>0.0051200</td>
<td>0.0051306</td>
<td>0.0051288</td>
<td>0.0051292</td>
<td>0.0051291</td>
</tr>
</tbody>
</table>

3 A determinantal reduction formula for continued fractions

Consider the continued fraction \( \frac{c_0}{z + \frac{c_1}{z + \frac{c_2}{z + \cdots}}} \) with convergents \( \frac{\chi_s}{\omega_s}, s = 1, 2, \ldots \). Then, for example,

\[
\chi_1 = c_0, \quad \chi_2 = zc_1,
\]

\[
\omega_1 = z, \quad \omega_2 = z + c_1. \quad (\chi_0 = 0, \omega_0 = 1).
\]

We may therefore consider the determinant

\[
\begin{vmatrix}
\chi_s & \chi_{s+1} \\
\omega_s & \omega_{s+1}
\end{vmatrix}
\]

using the recurrence formulas. Expanding the determinant leads to

\[
\frac{\chi_s}{\omega_s} = \frac{\chi_0}{\omega_0} + \left( \frac{\chi_1}{\omega_1} - \frac{\chi_0}{\omega_0} \right) + \cdots + \left( \frac{\chi_s}{\omega_s} - \frac{\chi_{s-1}}{\omega_{s-1}} \right) + \cdots
\]

at least formally. Hence

\[
\frac{\chi_s}{\omega_s} = \frac{c_0}{\omega_0 \omega_1} + \frac{c_0 c_1}{\omega_1 \omega_2} + \frac{c_0 c_1 c_2}{\omega_2 \omega_3}.
\]

Example 3
The Laplace continued fraction is

\[
\frac{1}{z+} \frac{1}{z+} \frac{2}{z+} \frac{3}{z+} \cdots \quad (z > 0)
\]
so \(\omega_1 = z\), \(\omega_2 = z^2 + z\), \(\omega_3 = z^3 + 3z\), \(\omega_4 = z^4 + 6z^2 + 3\). A simple formula is \(\omega_n = e^{-\frac{1}{2}d^2} x^2\) the polynomial being Hermite.

Now using (3) we may set up an expression for derivatives of a continued fraction using the Maple symbolic code.

4 Conclusion

We now pay attention to the function \(F(z)\) which represents the value to which the 2nd order continued fractions converges. If \(z\) is real and positive is \(F(z)\) positive?

For the first example we knew (Shenton and Bowman, 1971)

\[
F(z) = \frac{1}{2z} \left\{ \psi_1(z) - \frac{1}{z} - \frac{1}{2z^2} \right\}
= \frac{1}{2z} \frac{2\pi}{3} \int_0^\infty \frac{y\sqrt{x}dx}{xz^2(y-1)^2} \quad (y = e^{2\pi\sqrt{x}})
\]

which is positive for \(z > 0\).

Now consider the expression for \(F(z)\) arising from the continued fraction from \(\ln (\Gamma(z))\). In the sequel for this case

\[
F(z) = -\frac{1}{2z} \frac{d}{dz} \left\{ \frac{\ln(\Gamma(z))}{z} - \left(1 - \frac{1}{2z}\right) \ln z + 1 - \ln(2\pi) \right\}
= -\frac{1}{2z} \left\{ \psi(z) - \frac{\ln(\Gamma(z))}{z^2} - \ln z - \frac{1}{2z^2} \right\}
\]

We have been unable to prove this is positive, for \(z\) real and positive. So we test out several cases.

Case with \(z = \frac{1}{2}\):

\[
F(z) = \left\{ -2\psi(2) - 2\ln \pi - 2\ln \frac{1}{2} + 2\ln(2\pi) \right\} = 2\gamma \quad (\gamma = \text{Euler’s constant}).
\]

The upper bound is 4/3. It is quite remarkable that in the Handbook of Mathematical Functions.... section 6.3.3, we have the single entry (also see ”dlmf.nist.gov”, 5.4.13)

\[
\psi \left( \frac{1}{2} \right) = -\gamma - 2\ln 2 = 1.9635100260....
\]
As far as we can tell it is not referenced. What a remarkable piece of luck?

Now take \( F(1) \): We have

\[
F(1) = -\frac{1}{2} \left\{ -\gamma - \frac{1}{2} + \frac{\ln(2\pi)}{2} \right\} = 0.0058.
\]

The upper bound is 1/12.

For \( z = 2 \),

\[
F(2) = -\frac{1}{4} \left\{ \frac{-\psi(2)}{2} - \frac{\ln 2}{8} - \frac{3}{8} + \frac{\ln(2\pi)}{8} \right\}
\]
\[
= -\frac{1}{4} \left\{ \frac{1}{2}(1 - \gamma) - \frac{3}{8} + \frac{\ln(2\pi)}{8} \right\} = 0.0051 \ldots
\]

The upper bound is 0.0208.

References


