SECOND ORDER CONTINUED FRACTIONS AND FIBONACCI NUMBERS

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Abstract

Stieltjes transforms refer to integrals with integrand $d\sigma(t)/(t+z)$, the range from zero to ∞. Other continued fractions are related to this form and appear in the form $\frac{a_0}{z+}\frac{a_1}{1+}\frac{a_2}{z+}\frac{a_3}{1+}\cdots$. Second order continued fractions relate to a definite integral with integrand $d\sigma(t)/(t+z_1)(t+z_2)$. The Fibonacci c.f. is $\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots$ or $F(z)=\frac{1}{z+}\frac{1}{1+}\frac{1}{z+}\cdots$ with $z=1$. The 2nd order c.f., relates to $-dF/dz\,|_{z_1=z_2=1}$. Based on these 2nd order c.f.s., results are derived for properties of Fibonacci numbers and the squares of Fibonacci numbers.

Key words: Recurrence relations, S-fractions, Stieltjes integrals, Stieltjes moment problem.

1 Introduction

Fibonacci numbers 1,1,2,3,$\cdots$ are related to the continued fraction

$$F(z) = \frac{1}{z+}\ \frac{1}{1+}\ \frac{1}{z+}\ \frac{1}{1+}\ \cdots$$

where $z = 1$. In this case the c.f.

$$\frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{1+}\ \cdots \tag{1}$$

has convergents χ_s/ω_s, both of which follow a recurrence relation of order 2. For example

$$\omega_s = \omega_{s-1} + \omega_{s-2} \qquad (s = 2, 3, \cdots) \tag{2}$$

with $\omega_0 = 1$, $\omega_1 = 1$. The characteristic equation is $\lambda^2 - \lambda - 1 = 0$ with roots $\lambda_1 = (1+\sqrt{5})/2$, and $\lambda_2 = (1-\sqrt{5})/2$. In fact

$$\omega_s = \frac{\lambda_1^{s+1} - \lambda_2^{s+1}}{\sqrt{5}} \qquad (s = 0, 1, \cdots)$$

and $\lambda_1\lambda_2 = -1$.

How is the 2nd order recurrence related to that of the lst order c.f. given in (2), and properties of the ω_s denominator of (1). First of all a brief account of 2nd order c.f.s.

2 Second order c.f.s, basic formula

A continuant determinant has the form

$$\Delta_s = \begin{vmatrix} a_1 & b_1 & \cdots & & & \\ b_1^* & a_2 & b_2 & \cdots & & \\ 0 & b_2^* & a_3 & b_3 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & & & b_s^* & a_s \end{vmatrix}$$

with recurrence

$$\Delta_s = a_s\Delta_{s-1} - b_s^*\Delta_{s-2},$$

linear and of order two. This determinant is asymmetric with elements only in the main diagonal, the upper diagonal, and the lower diagonal. We may use a notation reflecting this property such as $K_s(a_s, b_s)$ when $b_s^* = b_s$ i.e. the symmetric case. A generalization is now obvious leading to a determinant of 5 diagonals, one central, 2 upper, and 2 lower. Recurrence relations for this case are considered in Shenton (1957, pp.167-188). We note in passing that Ursell (1958) considered the recurrence for a set of s linear equations, the coefficients being variable. As examples, he gave the following orders:

order 5(6)	for the 5 diagonal determinant
15(20)	for the 7 diagonal determinant
49(70)	for the 9 diagonal determinant
604(924)	for the 13 diagonal determinant

(Parenthetic entries refer to the asymptotic case).

It should be noted that although the order of a recurrence is known, it may be another matter to find it.

An encouraging practical application of c.f.s is to be found in papers by J.C. Shenton and L.R. Shenton (June 1960, November 1960, and September 1961) concerning harmonics in synchronous machines, and published by the Royal Aircraft Establishment (Farnborough).

The fundamental equation, a difference expression in Laplace transforms, this allowing the possible expansion by c.f.s. We find examples such as

$$\frac{a_n}{K} = \frac{a_1}{z+}\ \frac{a_2}{1+}\ \frac{a_3}{z+}\ \frac{a_4}{1+}\ \cdots$$

where

$$a_{2s} = 1/g_s(n+2s-2), \qquad a_{2s+1} = k^2/g_s(n+2s);$$

also

$$\beta_n = \frac{1}{1-it_n-}\ \frac{k^2}{1-it_{n+1}-}\ \frac{k^2}{1-it_{n+2}-}\ \cdots \qquad (n = 1, 2, \cdots)$$

3 Second order c.f.s. and definite integrals

A Stieltjes transform may be associated with a c.f. Thus

$$F(z) = \int_0^\infty \frac{d\psi(t)}{t+z}, \qquad (\Re(z) > 0)$$

$$= \frac{b_1}{z+} \; \frac{b_2}{1+} \; \frac{b_3}{z+} \; \frac{b_4}{z+} \; \cdots,$$

and its even part

$$\frac{a_0}{z + c_1 -} \; \frac{a_1}{z + c_2 -} \; \frac{a_2}{z + c_3 -} \; \cdots$$

A second order c.f. is defined by

$$F(z_1, z_2) = \int_0^\infty \frac{d\sigma(t)}{(t + z_1)(t + z_2)}$$

if it exists. The approximants are k_s^* and k_s where

$$F(z_1, z_2) = \lim_{s \to \infty} \frac{k_s^*}{k_s},$$

and

$$k_s^* = K_{s-1}(\gamma_1, \beta_1, \alpha_1), \quad k_s = K_s(\gamma_0, \beta_0, \alpha_0)$$

these being 5 diagonal determinants, γ in the central diagonal, β in the first upper diagonal and first lower diagonal, α in the second upper diagonal and second sub-diagonal, the orders being denoted by the subscripts s in K_s.

$$\begin{aligned}
\alpha_s &= \sqrt{a_{s+1} a_{s+2}}, \\
\beta_s &= (p + c_{s+1} + c_{s+2}) \sqrt{a_{s+1}}, \\
\gamma_s &= q + p c_{s+1} + c_{s+1}^2 + a_s^* + a_{s+1}, \\
a_s^* &= a_s, s > 0, \quad a_0^* = 0, \\
(t + z_1)(t + z_2) &\equiv t^2 + pt + q > 0, \quad x \geq 0.
\end{aligned}$$

There is a similar form for a "third order" c.f. associated with the Stieltjes integral

$$F(z_1, z_2, z_3) = \int_0^\infty \frac{d\sigma(t)}{(t + z_1)(t + z_2)(t + z_3)}$$

with the assumption of existence. (Shenton, 1957, p.168)

4 A basic theorem

In Shenton (1957) there is the theorem concerning a S-fraction, namely, if the Stieltjes moment problem is determined and

$$F(z) = \int_0^\infty \frac{d\psi(t)}{t+z} = \frac{b_1}{z+} \; \frac{b_2}{1+} \; \frac{b_3}{z+} \; \frac{b_4}{1+} \; \frac{b_5}{z+} \; \frac{b_6}{1+} \; \cdots, \tag{3}$$

then

$$l.i.s._{s\to\infty} \frac{k_{2s}^*}{k_{2s}} = \lim_{s\to\infty} \frac{k_s^*}{k_s} = \frac{F(z_2) - F(z_1)}{z_1 - z_2}, \qquad (z_1 \neq z_2)$$

where

- k_s^* and k_s follow, for $s = 2, 3, \cdots$,
  $W_{2s-1} = z_1 z_2 W_{2s-2} + \alpha_{2s-1} W_{2s-3} - \beta_{2s-1} W_{2s-5} - z_1 z_2 \gamma_{2s-1} W_{2s-6} + \delta_{2s-1} W_{2s-7}$,
  $W_{2s} = W_{2s-1} + \alpha_{2s} W_{2s-2} - \beta_{2s} W_{2s-4} - \gamma_{2s} W_{2s-5} + \delta_{2s} W_{2s-6}$.

- $k_0^* = 0$, $k_1^* = k_2^* = b_1$, $\qquad k_s^* = 0$, $s < 0$,
  $k_0 = 1$, $k_1 = z_1 z_2$, $k_2 = z_1 z_2 + b_2(z_1 + z_2 + b_3 + b_2)$, $\qquad k_s = 0$, $s < 0$,

- $\alpha_s = b_s(z_1 + z_2 + b_{s+1} + b_s + b_{s-1})$, $\quad \beta_s = b_s b_{s-2} \alpha_{s-1}$,
  $\gamma_s = b_s b_{s-1} b_{s-2} b_{s-3}$, $\quad \delta_s = b_s b_{s-1} b_{s-2}^2 b_{s-3} b_{s-4}$;

- $(t + z_1)(t + z_2) > 0$ for $x \geq 0$.

It may be established that 'odd' part k_{2s-1}^*/k_{2s-1} arises from the second order c.f. associated with the integral in the expression

$$z_1 z_2 \left(\frac{F(z_2) - F(z_1)}{z_1 - z_2} \right) = b_1 - \int_0^\infty \frac{t(t + z_1 + z_2) d\psi(t)}{(t + z_1)(t + z_2)}$$

where $t d\psi(t)$ is taken as the weight function. The 'odd' part of the sequence, unlike that of a Stieltjes c.f., does not in general provide a set of decreasing upper bounds.

5 A recurrence for ω_s $(\omega_0 = 1, \omega_1 = 1, \omega_2 = 2, \omega_3 = 3)$

From (1) ω_s is a sth Fibonacci number and

$$\omega_{2s}^2 = (3\omega_{2s-2} - \omega_{2s-4})^2 = 9\omega_{2s-2}^2 + \omega_{2s-4}^2 - 6\omega_{2s-2}\omega_{2s-4}. \quad (s = 2, 3, \cdots) \quad (4)$$

But $\omega_{2s-2}\omega_{2s-4} = \omega_{2s-4}(3\omega_{2s-4} - \omega_{2s-6})$ and substituting from (4) we have the recurrence

$$\omega_{2s}^2 - 8\omega_{2s-2}^2 + 8\omega_{2s-4}^2 - \omega_{2s-6}^2 = 0 \quad (s = 3, \cdots) \quad (5)$$

Let

$$\Phi_{2s} = \omega_{2s}^2 - 7\omega_{2s-2}^2 + \omega_{2s-4}^2. \quad (s > 1)$$

Then from equation (5)

$$\Phi_{2s} - \Phi_{2s-2} = 0 \qquad (s = 2, \cdots)$$

Thus

$$\Phi_{2s} = \Phi_2 = 5^2 - 7 \cdot 2^2 + 1 = -2.$$

Hence

$$\omega_{2s}^2 - 7\omega_{2s-2}^2 + \omega_{2s-6}^2 = -2. \quad (s = 3, \cdots)$$

Similarly if

$$\Phi_{2s+1} = \omega_{2s+1}^2 - 7\omega_{2s-1}^2 + \omega_{2s-3}^2$$

along with

$$\omega_{2s+1}^2 - 8\omega_{2s-1}^2 + 8\omega_{2s-3}^2 - \omega_{2s-5}^2 = 0 \quad (s = 3, \cdots)$$

then

$$\Phi_{2s+1} = \Phi_{2s-1},$$

leading to

$$\Phi_{2s+1} = \Phi_3 = (21)^2 - 7 \cdot 8^2 + 3^2 = 2.$$

Together, then

$$\begin{aligned} \omega_{2s}^2 - 7\omega_{2s-2}^2 + \omega_{2s-4}^2 &= -2 \quad (s = 2, \cdots) \\ \omega_{2s+1}^2 - 7\omega_{2s-1}^2 + \omega_{2s-3}^2 &= +2 \quad (s = 2, \cdots) \end{aligned} \quad (6)$$

By a completely different approach, Rajesh (2004) has provided the first of the equations. The equation (6) may be regarded as a number theorem problem. For example find solutions in integer (> 0) such that

$$n_1^2 - 7n_2^2 + n_3^2 = \pm 2,$$

a type of Diophantine analysis. Note that the zeros of $x^2 - 7x + 1$ are $x_1 = (7+3\sqrt{5})/2$, and $x_2 = (7-3\sqrt{5})/2$ and in terms of Fibonacci parameters

$$x_1 = \left(\frac{1+\sqrt{5}}{2}\right)^4, \qquad x_2 = \left(\frac{1-\sqrt{5}}{2}\right)^4.$$

6 Second order c.f.s, simultaneous recurrences

6.1 Details of the basic theorem

Returning to the basic theorem in §4, we take $z_1 = 1$, $z_2 = 1$ so that (3) becomes

$$F(z) = \frac{1}{z+}\ \frac{1}{1+}\ \frac{1}{z+}\ \frac{1}{1+}\ \cdots$$

and $F(1)$ refers to the Fibonacci case. Note that when $z_1 = z_2$ we are considering

$$-\frac{d}{dz}\int_0^\infty \frac{d\psi(t)}{t+z} \quad \text{or} \quad -\frac{d}{dz}\frac{1}{2\pi}\int_0^4 \frac{\sqrt{4t^{-1}-1}}{t+z}dt$$

where

$$F(z) = -\frac{1}{2} + \frac{\sqrt{1+4/z}}{2} \qquad (z > 0)$$

so that the derivative is

$$F'(z) = -\frac{4/z^2}{4\sqrt{1+4/z}},$$

and in particular $-F'(1) = 1/\sqrt{5}$.

The second order c.f. sequences are derived from (k_s^*) and (k_s) where these follow the recurrences

$$\begin{aligned} W_{2s-1} &= W_{2s-2} + 5W_{2s-3} - 5W_{2s-5} - W_{2s-6} + W_{2s-7}, \\ W_{2s} &= W_{2s-1} + 5W_{2s-2} - 5W_{2s-4} - W_{2s-5} + W_{2s-6} \end{aligned} \tag{7}$$

with $k_0^* = 0$, $k_1^* = k_2^* = 1$, $k_0 = 1$, $k_1 = 1$, $k_2 = 5$, with $k_s^* = 0$, $s < 0$, $k_s = 0$, $s < 0$. Further values are given in Table 1. The approximants k_s^*/k_s form an enveloping sequence (see Shenton, 1957, p184), showing that even convergents k_{2s}^*/k_{2s} form a monotonic increasing set of lower bounds, whereas the sequence k_{2s+1}^*/k_{2s+1} forms a monotonic decreasing set of upper bounds. This characteristic is analogous to the situation for first order c.f. such as $\frac{b_1}{1+}\frac{b_2}{1+}\frac{b_3}{1+}\cdots$ provided b_1, b_2, b_3 $\cdots$ are positive values.

Table 1. k_s^* and k_s for approximants to $1/\sqrt{5}$, a second order c.f.

s	k_s^*	k_s	t_s	s	k_s^*	k_s	t_s
1	1	1	1.0	11	10866	24276	0.4476
2	1	5	0.2	12	28416	63565	0.4470
3	6	10	0.6	13	74431	166405	0.447288
4	11	30	0.37	14	194821	435665	0.447181
5	36	74	0.49	15	510096	1140574	0.447227
6	85	199	0.43	16	1385395	2986074	0.447208
7	235	515	0.456	17	3496170	7817630	0.447218
8	600	1355	0.443	18	9153025	20466835	0.4472125
9	1590	3540	0.449	19	23963005	53582855	0.4472140
10	4140	9276	0.4463	20	62735880	140281751	0.4472134
				∞	——	——	0.4472136

We have

$$\lim_{s\to\infty}\left(\frac{k_s^*}{k_s}\right) = \frac{1}{\sqrt{5}}.$$

6.2 Expression for $k_s - k_{s-2}$

From Shenton (1957, p.180), we have

$$\left\{\begin{aligned} k_{2s} - k_{2s-2} &= \omega_{2s}^2 \\ k_{2s+1} - k_{2s-1} &= \omega_{2s-1}^2 \end{aligned}\right. \tag{8}$$

since $b_1 = b_2 = b_3 = 1$. Using (6), these lead to the recurrences (non-homogeneous)

$$\left\{\begin{aligned} k_{2s} - 8k_{2s-2} + 8k_{2s-4} - k_{2s-6} &= -2 \quad (s = 3, \cdots) \\ k_{2s+1} - 8k_{2s-1} + 8k_{2s-3} - k_{2s-5} &= +2 \quad (s = 3, \cdots) \end{aligned}\right. \tag{9}$$

Note that for the homogeneous part of (7), the characteristic equation

$$t^3 - 8t^2 + 8t - 1 = 0,$$

has roots given by

$$(t-1)(t^2 - 7t + 1) = 0,$$

or $t_1 = 1$, $t_2 = \left(\frac{1+\sqrt{5}}{2}\right)^4$, $t_3 = \left(\frac{1-\sqrt{5}}{2}\right)^4$ suggesting a solution of the form

$$A_0 + A_1 s + A_2 \left(\frac{1+\sqrt{5}}{2}\right)^{4s} + A_3 \left(\frac{1-\sqrt{5}}{2}\right)^{4s}.$$

where $A_3 = \pm A_3$.

6.3 k_{2s} as a Fibonacci number

From equation (8)

$$\begin{cases} k_{2s} - k_{2s-2} = \omega_{2s}^2 \\ k_{2s-2} - k_{2s-4} = \omega_{2s-2}^2 \\ \vdots \quad \vdots \qquad \vdots \\ k_2 - k_0 = \omega_2^2 \end{cases}.$$

By addition

$$\begin{aligned} 5(k_{2s} - k_0) &= (\lambda_1^{2s+1} - \lambda_2^{2s+1})^2 + (\lambda_1^{2s-1} - \lambda_2^{2s-1})^2 + \cdots + +(\lambda_1^3 - \lambda_2^3) \\ &= 2s + \lambda_1^6(\lambda_1^{4s-4} + \lambda_1^{4s-8} + \cdots + 1) + \lambda_2^6(\lambda_2^{4s-4} + \lambda_2^{4s-8} + \cdots + 1) \\ &= 2s + \frac{\lambda_1^6(\lambda_1^4 - 1)}{\lambda_1^4 - 1} + \frac{\lambda_2^6(\lambda_2^4 - 1)}{\lambda_2^4 - 1}. \end{aligned}$$

But

$$\lambda_1^4 - 1 = (\lambda_1^2 - 1)(\lambda_1^2 + 1) = \lambda_1(\lambda_1 + 2) = \lambda_1^2\sqrt{5},$$

since $\lambda_1^2 = \lambda_1 + 1$. Similarly,

$$\lambda_2^4 - 1 = (\lambda_2^2 - 1)(\lambda_2^2 + 1) = \lambda_2(\lambda_2 + 2) = -\lambda_2^2\sqrt{5},$$

Hence

$$5(k_{2s} - k_0) = 2s + \omega_{4s+3} - \omega_3.$$

so

$$5k_{2s} = 2(s+1) + \omega_{4s+3}. \qquad (s = 0, 1, \cdots) \tag{10}$$

Similarly using (8) for k_{2s+1} we find the relation

$$5k_{2s+1} = -2s - 3 + \omega_{4s+5}. \qquad (s = 0, 1, \cdots)$$

Examples

k_{2s} :

$$\begin{aligned} 5k_0 &= 2 + 3 = 5 \\ 5k_2 &= 4 + 21 = 25 \\ 5k_4 &= 6 + 144 = 150 \\ 5k_{10} &= 12 + \omega_{23} = 12 + 46368 = 46380, \end{aligned}$$

k_{2s+1} :

$$\begin{aligned} 5k_1 &= -3 + 8 = 5 \\ 5k_5 &= -7 + 377 = 370, \quad k_5 = 74 \\ 5k_7 &= -9 + 2584 = 2575, \quad k_7 = 515 \\ 5k_9 &= -11 + \omega_{21} = -11 + 17711 = 17700, \quad k_9 = 3540 \end{aligned}$$

6.4 k^*_{2s+1} in terms of ω_s^2

A study of Table 1 provides numeral evidence that

$$k^*_{2s+1} - k^*_{2s-1} = k_2 s.$$

A proof of this is given in an Appendix. We have, from (10),

$$\left\{ \begin{array}{c} 5(k^*_{2s+1} - k^*_{2s-1}) = 2(s+1) + \omega_{4s+3}, \\ 5(k^*_{2s-1} - k^*_{2s-3}) = 2s + \omega_{4s-1}, \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ 5(k^*_3 - k^*_1) = 2^2 + \omega_7. \end{array} \right.$$

By addition

$$5(k^*_{2s+1} - k^*_1) = 2\left(\frac{(s+1)(s+4)}{2} - 1\right) + \Psi_s$$

where

$$\begin{aligned}
\Psi_s &= \frac{\lambda_1^{4s+4} - \lambda_2^{4s+4}}{\sqrt{5}} + \frac{\lambda_1^{4s} - \lambda_2^{4s}}{\sqrt{5}} + \cdots + \frac{\lambda_1^{8} - \lambda_2^{8}}{\sqrt{5}} \\
&= \frac{(\lambda_1^{4s+4} + \lambda_1^{4s} + \cdots + \lambda_1^{8})}{\sqrt{5}} - \frac{(\lambda_2^{4s+4} + \lambda_2^{4s} + \cdots + \lambda_2^{8})}{\sqrt{5}} \\
&= \frac{\lambda_1^8(\lambda_1^{4s} - 1)}{\sqrt{5}(\lambda_1^4 - 1)} - \frac{\lambda_2^8(\lambda_2^{4s} - 1)}{\sqrt{5}(\lambda_2^4 - 1)},
\end{aligned}$$

where $\lambda_1^4 - 1 = \lambda_1^2\sqrt{5}$, and $\lambda_2^4 - 1 = -\lambda_2^2\sqrt{5}$. Hence

$$\begin{aligned}
\Psi_s &= \frac{\lambda_1^6(\lambda_1^{4s} - 1)}{5} + \frac{\lambda_2^6(\lambda_2^{4s} - 1)}{5}, \\
&= \left(\frac{\lambda_1^{2s+3} - \lambda_2^{2s+3}}{\sqrt{5}}\right)^2 - \left(\frac{\lambda_1^{3} - \lambda_2}{\sqrt{5}}\right)^2 \\
&= \omega_{2s+2}^2 - \omega_2^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
5k^*_{2s+1} &= 5 + (s-1)(s+2) - 2 - 4 + \omega_{2s-1}^2, \\
&= s^2 + 3s + 1 + \omega_{2s+2}^{\text{¡}}
\end{aligned}$$

a strange form compared to the case for $5k_{2s}$ and k_{2s+1} given in (10).

6.5 k^*_{2s} in terms of ω_s^2

Again from the appendix

$$k^*_{2s} - k^*_{2s-2} = k_{2s-1}.$$

This leads to the scheme:

$$\begin{cases}
5(k^*_{2s} - k^*_{2s-2}) = -2s - 1 + \omega_{4s+1} \\
5(k^*_{2s-2} - k^*_{2s-4}) = -2s - 1 + \omega_{4s-3} \\
5(k^*_{2s-4} - k^*_{2s-6}) = -2s - 1 + \omega_{4s-7} \\
5(k^*_{4} - k^*_{2}) = -5 + \omega_{9}.
\end{cases}$$

Adding,

$$\begin{aligned}5(k_{2s}^*-k_2^*) &= -(s+1)(s+3)+\frac{\lambda_1^{4s+2}-\lambda_2^{4s+2}}{\sqrt{5}}+\frac{\lambda_1^{4s-3}-\lambda_2^{4s-3}}{\sqrt{5}}+\cdots+\frac{\lambda_1^{10}-\lambda_2^{10}}{\sqrt{5}}\\ &= -(s+1)(s+3)+\frac{\lambda_1^{10}(\lambda_1^{4s-4}-1)}{\sqrt{5}(\lambda_1^4-1)}+\frac{\lambda_2^{10}(\lambda_2^{4s-4}-1)}{\sqrt{5}(\lambda_2^4-1)},\end{aligned}$$

so finally

$$5k_{2s}^* = -(s+1)^2+\omega_{2s+1}^2.$$

Example of $5k_{2s}^*$.

$$\begin{aligned}
&s=0; \quad 5k_0^* = -1+1=0, \quad k_0^*=0\\
&s=1; \quad 5k_2^* = -4+9=5, \quad k_2^*=1\\
&s=2; \quad 5k_4^* = -9+64=55, \quad k_4^*=11\\
&s=3; \quad 5k_6^* = -16+441=425, \quad k_6^*=85
\end{aligned}$$

7 Conclusion

Properties of Fibonacci numbers are derived based on the second order c.f. corresponding to $F(z)=\frac{1}{z+}\,\frac{1}{1+}\,\frac{1}{z}\,\cdots$, the second order c.f. refers to $-\frac{dF}{dz}\,|_{z=1}$. It is therefore possible that series of the results may be new.

Note that, for example

$$\frac{5k_{2s}^*}{5k_{2s}} = \frac{-(s+1)^2+\omega_{2s+1}^2}{2(s+1)+\omega_{4s+3}},$$

so as $s\to\infty$,

$$\frac{k_{2s}^*}{k_{2s}} \sim \frac{\omega_{2s+1}^2}{\omega_{4s+3}} \sim \frac{(\lambda_1^{2s+2}/\sqrt{5})^2}{\lambda_1^{4s+4}/\sqrt{5}};$$

i.e. the limits is $1/\sqrt{5}$, as it should. A similar result holds for $\lim_{s\to\infty}\frac{k_{2s+1}^*}{k_{2s+1}}$.

Appendix

A1. Concerning $k^*_{2s} - k^*_{2s-2}$ and $k^*_{2s+1} - k^*_{2s-1}$

From Shenton (1957, p174, equation (18)) we have, adjusting the notation to our present study

$$k^*_{2s} = 10k^*_{2s-2} - 25k^*_{2s-4} + 25k^*_{2s-6} - 10k^*_{2s-8} + k^*_{2s-10} \quad (s = 5, \cdots) \tag{11}$$

and

$$k^*_{2s-2} = 10k^*_{2s-4} - 25k^*_{2s-6} + 25k^*_{2s-8} - 10k^*_{2s-10} + k^*_{2s-12} \quad (s = 6, \cdots)$$

Subtract and deduce an expression for $k^*_{2s} - k^*_{2s-2}$ on the left side; the right hand side becomes,

$$10k_{2s-3} - 25k_{2s-5} + 25k_{2s-7} - 10k_{2s-9} + k_{2s-11} \tag{12}$$

if the assumption in (11) is correct. But (11) is also true for k_s, the only difference being in initial values. Hence (12) predicts the value k_{2s-1}. This approach provides an inductive proof for the assumption in (A1).

A2. A linear equation for ω_s

From equation (8), elimination leads to

$$k_{2s} - 9k_{2s-2} + 16k_{2s-4} - 9k_{2s-6} + k_{2s-8} = 0, \quad (s = 2, 4, \cdots)$$

and similarly for k_{2s+1}. But, for example,

$$k_{2s} = 2(s+1) + \omega_{4s+3} \qquad (s = 0, 1, \cdots)$$

In this way, we can set up the linear relations for Fibonacci numbers,

$$\omega_{4s+19} - 9\omega_{4s+15} + 16\omega_{4s+11} - 9\omega_{4s+7} + \omega_{4s+3} = 0 \quad (s = 0, 1, \cdots)$$

$$\omega_{4s+17} - 9\omega_{4s+13} + 16\omega_{4s+9} - 9\omega_{4s+5} + \omega_{4s+1} = 0. \quad (s = 0, 1, \cdots)$$

A3. ω_s and divisibility by 5

Consider equations such as

$$\begin{aligned}
5k_{2s} &= 2(s+1) + \omega_{4s+3}, \\
5k_{2s+1} &= -2s - 3 + \omega_{4s+5}, \\
5k^*_{2s+1} &= s^2 + 3s + 1 + \omega^2_{2s+2}, \\
5k^*_{2s} &= -(s+1)^2 + \omega^2_{2s+1}.
\end{aligned}$$

These will be divisibility by 5 if the first term on the right is divisible by 5. For example, $(s+1)^2$ is divisible by 5 if $s = 4,\ 9,\ 14, \cdots$. Checking each case in this way leads to a proof that ω_s, the Fibonacci number, is divisible by 5 for $s = 4,\ 9,\ 14,\ \cdots$.

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References

Bowman, K.O. and Shenton, L.R. (1989). *Continued Fractions in Statistical Applications*, Marcel Dekker, Inc.

Comtet, Louis (1974). *Advanced Combinatories*, D. Reidel Pub. Co., Boston, USA.

Rajesh, V. (2004). Some properties of odd terms of the Fibonacci sequence, *The Mathematical Gazette*, 88(51), p.85.

Shenton, J.C. and Shenton, L.R. (1960). *Harmonics in Synchronous Machines on Single Phase Line-to-line resistance Load.* , Royal Aircraft Establishment Ref.EL.3902. (Pages 1-2, 24 figures).

Shenton, J.C. and Shenton, L.R. (1960). *Harmonics in Synchronous Machines. Part 2. Three Phase Short Circuit*, $r_a > 0$, $r_b = r_c = 0$, Royal Aircraft Establishment Ref.EL.1847. (Pages 1-39, 10 figures).

Shenton, J.C. and Shenton, L.R. (1961). *Harmonics in Synchronous Machines. Part 3. Three Phase Short Circuit*, $r_a > 0$, $r_b = r_c =\neq 0$, Royal Aircraft Establishment Ref.EL.1855. (Pages 1-50, 38 figures).

Shenton, L.R. (1957). A determinantal expansion for a class of recurrence relations. *Edinb.Math.Proc.*, 10(IV), 167-188.

Ursell, H.D. (1958). Simultaneous linear recurrence relations with variable coefficients, *Proc.Edinb.Math.soc.*, 9(4), 183-206.