Adaptive Discontinuous Galerkin Methods with Multiwavelets Bases

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Abstract

We demonstrate the advantages of using multi-reolution analysis with multiwavelet basis with the Discontinuous Galerkin (DG) method. This provides significant enhancements to the standard DG methods. To illustrate the important gains of using the Multiwavelet DG method we apply it to conservation and convection diffusion problems in multiple dimensions. The significant benefits of merging DG methods with multiwavelets are three-fold. First, the DG method inherits a hierarchical structure from multiwavelets that produces a weak decoupling across different length scales. Second, hp-adaptivity in the DG method is naturally resolved through the multiwavelet basis rather than grid manipulation by the scaling properties of multiwavelets. Third, the matrix of the multiwavelet DG operator and its inverse share the same sparse pattern, that has the potential to provide nearly linear scaling of memory and computational performance with increasing degrees of freedom and dimensionality. In addition, the highly desired sparsity pattern combined with multiresolution provides a direct way for developing fast numerical solvers. These properties are especially important for higher dimensional problems with large degrees of freedom.

Key Words: Multiwavelets, Discontinuous Galerkin

1 Introduction

The discontinuous Galerkin (DG) method is a type of finite element method that is locally conservative and stable with high-order accuracy. The DG formulation utilizes an element-wise discontinuous approximation, where numerical information only passes locally through numerical fluxes, to accurately represent the dynamics and structure of highly complex solutions. Additionally, the elegant and flexible formulation of DG allows this method to handle complex geometries, irregular meshes with hanging nodes, hp-adaptation, scalability, and solution approximations that may have different basis functions of different order in different elements. All of these properties have made DG an attractive numerical method for various engineering and scientific problems. We refer the reader to [13], and references therein, for an excellent description of the DG method and its applications.

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Along a similar time-line as DG has been the development of multiwavelets, which has not only produced a multiresolution basis, but also fast adaptive solver methods for both linear and nonlinear partial differential equations [2]. As a bases, multiwavelets are a discontinuous, orthogonal, compactly supported, multiscale set of functions with vanishing moments that yield high-order hp-adaptive approximations of L^2 functions. With respect to integrodifferential operators, the properties of multiwavelets has made them an attractive bases that can provide an effective sparse represention of a wide class of integrodifferential operators.

This research combines the structure of multiwavelets with the DG method, and demonstrates how this union results in an efficient multi-scale adaptive DG method. We refer to this new approach as multiwavelet DG, which has the following advantages. First, multiwavelet DG has a hierarchical structure that produces a weak decoupling across different length scales. Second, hp-adaptivity in multiwavelet DG is computationally fast and effective with a well defined structure. Third, the matrix of the multiwavelet DG operator and its inverse share the same sparse pattern. In addition, the highly desired sparsity pattern combined with multiresolution provides a direct way for developing fast numerical solvers. These properties are especially important for higher dimensional problems with large degrees of freedom.

This paper is orgainized as follows. In Section 2 we introduce the multiwavelet bases and its key features. Section 3 and 4 describe the DG method for conservation and convection diffusion problems, respectively, and further demonstrate how multiwavelet bases can be incorporated into these methods. Section 5 describes the method of time stepping used in this research, which has been demonstrated to be particularly effective and efficient for multiwavelet based schemes [2, 5]. Examples in multiple dimensions and different geometry are given in Section 6 for both conservation and convection diffusion problems. Section 7 finishes this paper with conclusions and a discussion of future directions.

2 Multiwavelet Bases

In this section we summarize some properties of the multiwavelet bases derived and developed in [1] and introduce notation as given in [2]. We begin by reviewing the spaces associated with the multiwavelets, and describe the important two scale relationship. We finish this section with the form of the multiwavelet approximation and demonstrate how thresholding can be applied to the multiwavelet approximation.

For $k = 1, 2, \ldots,$ and $n = 0, 1, 2, \ldots,$ we define $\mathbf{V}_n^k$ as a space of piecewise polynomial functions,

$$\mathbf{V}_n^k = \{f : f \in \Pi_k(I_{nl}), \text{ for } l = 0, ..., 2^n - 1, \text{ and } supp(f) = I_l^n\}, \tag{2.1}$$

where $\Pi_k(I_l^n)$ is the space of all polynomial of degree less than k on the interval $I_l^n = [2^{-n}l, 2^{-n}(l+1)]$. The space $\mathbf{V}_n^k$ has dimension $2^n k$ and has the following nested property,

$$\mathbf{V}_0^k \subset \mathbf{V}_1^k \subset \cdots \subset \mathbf{V}_n^k \subset \cdots. \tag{2.2}$$

The multiwavelet subspace $\mathbf{W}_n^k$, $n = 0, 1, 2, \ldots,$ is defined as the orthogonal complement of $\mathbf{V}_n^k$ in $\mathbf{V}_{n+1}^k$ or

$$\mathbf{V}_n^k \oplus \mathbf{W}_n^k = \mathbf{V}_{n+1}^k, \quad \mathbf{W}_n^k \perp \mathbf{V}_n^k, \tag{2.3}$$

where the norm is defined as $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$, with

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx. \tag{2.4}$$

From the previous definition, we can split $\mathbf{V}_n^k$ into $n+1$ orthogonal subspaces as,

$$\mathbf{V}_n^k = \mathbf{V}_0^k \oplus \mathbf{W}_0^k \oplus \mathbf{W}_1^k \oplus \ldots \oplus \mathbf{W}_{n-1}^k. \tag{2.5}$$

Given a basis $\phi_0, \ldots, \phi_{k-1}$ of $\mathbf{V}_0^k$, the space $\mathbf{V}_n^k$ is spanned by $2^n k$ functions which are obtained from $\phi_0, \ldots, \phi_{k-1}$ by dilation and translation,

$$\phi_{jl}^n(x) = 2^{n/2}\phi_j(2^n x - l), \quad j = 0, \ldots, k-1, \quad l = 0, \ldots, 2^n - 1. \tag{2.6}$$

By construction similar properties hold for multiwavelets. If the piecewise polynomial functions, $\psi_0, ..., \psi_{k-1}$ form an orthonormal basis for $\mathbf{W}_0^k$, then by dilation and translation the space $\mathbf{W}_n^k$ is spanned by $2^n k$ functions

$$\psi_{jl}^n = 2^{n/2}\psi_j(2^n x - l), \quad j = 0, \ldots, k-1, \quad l = 0, \ldots, 2^n - 1. \tag{2.7}$$

The relations (2.2) and (2.3) between the subspaces may be expressed by the two-scale difference equations,

$$\begin{aligned}
\phi_i(x) &= \sqrt{2}\sum_{j=0}^{k-1}(h_{ij}^{(0)}\phi_j(2x) + h_{ij}^{(1)}\phi_j(2x-1)), \quad i = 0, ..., k-1,\\
\psi_i(x) &= \sqrt{2}\sum_{j=0}^{k-1}(g_{ij}^{(0)}\phi_j(2x) + g_{ij}^{(1)}\phi_j(2x-1)), \quad i = 0, ..., k-1,
\end{aligned} \tag{2.8}$$

where the coefficients can be determined through Gauss-Legendre quadrature, by multiplying and integrating Equations 2.8 by specific orthogonal basis functions to obtain,

$$\begin{aligned}
h_{ij}^{(0)} &= \frac{1}{\sqrt{2}}\sum_{m=0}^{k-1} w_m \phi_i\Big(\frac{x_m}{2}\Big)\phi_j(x_m),\\
h_{ij}^{(1)} &= \frac{1}{\sqrt{2}}\sum_{m=0}^{k-1} w_m \phi_i\Big(\frac{x_m+1}{2}\Big)\phi_j(x_m),\\
g_{ij}^{(0)} &= \frac{1}{\sqrt{2}}\sum_{m=0}^{k-1} w_m \psi_i\Big(\frac{x_m}{2}\Big)\phi_j(x_m),\\
g_{ij}^{(1)} &= \frac{1}{\sqrt{2}}\sum_{m=0}^{k-1} w_m \psi_i\Big(\frac{x_m}{2}\Big)\phi_j(x_m),
\end{aligned} \tag{2.9}$$

for the Gauss-Legendre nodes, $x_0, \ldots, x_{k-1}$, and weights, $w_0, \ldots, w_{k-1}$. For this study the Legendre polynomials $P_0, ..., P_{k-1}$ are used on (-1, 1), to construct an orthonormal basis for $\mathbf{V}_0^k$. Specifically, for $j = 0, ..., k-1$, we define the scaling functions as

$$\phi_j(x) = \begin{cases} \sqrt{2j+1}P_j(2x-1), & \text{if } x \in (0,1),\\ 0, & \text{otherwise.}\end{cases} \tag{2.10}$$

We use the algorithm given in [1] to construct the wavelets basis with the choice of vanishing moments.

Given a function, $f \in \mathbf{V}_n^k$, it can be represented by an expansion of scaling functions

$$f_h(x) = \sum_{l=0}^{2^n-1} \sum_{j=0}^{k-1} s_{jl}^n \phi_{jl}^n(x), \tag{2.11}$$

where the coefficients s_{jl}^n are

$$s_{jl}^n = \int_{2^{-n}l}^{2^{-n}(l+1)} f(x)\phi_{jl}^n(x)dx. \tag{2.12}$$

The function $f(x)$ has an multiwavelet expansion given by

$$f_h(x) = \sum_{j=0}^{k-1} \left(s_{j0}^0 \phi_j(x) + \sum_{m=0}^{n-1} \sum_{l=0}^{2^m-1} d_{jl}^m \psi_{jl}^m(x) \right), \tag{2.13}$$

with the coefficients

$$d_{jl}^m = \int_{2^{-n}l}^{2^{-n}(l+1)} f(x)\psi_{jl}^n(x)dx. \tag{2.14}$$

Using the two-scale difference Equations 2.8, the following relations between the coefficients on two consecutive levels m and $m+1$ can be generated,

$$\begin{aligned} s_{jl}^m &= \sum_{j=0}^{k-1} \left(h_{ij}^{(0)} s_{j,2l}^{m+1} + h_{ij}^{(1)} s_{j,2l+1}^{m+1} \right), \\ d_{jl}^m &= \sum_{j=0}^{k-1} \left(g_{ij}^{(0)} s_{j,2l}^{m+1} + g_{ij}^{(1)} s_{j,2l+1}^{m+1} \right), \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} s_{j,2l}^{m+1} &= \sum_{j=0}^{k-1} \left(h_{ji}^{(0)} s_{j,l}^m + g_{ji}^{(0)} d_{jl}^m \right), \\ d_{j,2l+1}^{m+1} &= \sum_{j=0}^{k-1} \left(h_{ji}^{(1)} s_{j,l}^m + g_{ji}^{(1)} d_{j,l}^m \right). \end{aligned} \tag{2.16}$$

These equations are used to create fast transforms between Equations 2.11 and 2.13.

Next, we define the truncation procedure that is used in this paper. We follow the approach given in [4], where for a give error tolerance level ϵ, the coefficient in the expansion in Equation 2.13 are found to produce an error of $\|f - f_h\|_\infty$. The truncation procedure is base on the property that $I_l^n = I_{2l}^{n+1} \bigcup I_{2l+1}^{n+1}$. Specifically, an extra level of refinement occurs on I_l^n if $\max_{x \in I_l^n} |f(x) - f_h(x)| > \epsilon$, where the maximum is approximated by considering the Gaussian nodes in the interval I_l^n.

(a) (b)

Figure 2.1: (a) First five hierarchical basis functions on the interval $[0,1]$. (b) Next level of refinement with nine hierarchical basis functions.

At first glance, Multiwavelets have a similarity to hierarchical basis functions [3], which have been applied in finite element and volume methods. Figure 2.1(a) depicts the first five hierarchical basis functions on a uniform grid on the interval $[0,1]$. Figure 2.1(b) depicts the next level of refinement for the hierarchical basis, where more hat functions of increasing smaller support are added. Figure 2.2(a) depicts the error in the approximation of the function, $f(x) = \sin(\pi x)$, using all 1025 hierarchical basis. At the nodes, the hierarchical basis function reaches machine precision, but off the nodes the reconstruction is significantly worse. In contrast, Figure 2.2(b) depicts the error in the approximation of the same function using multiwavelets for $k = 4$. With a basis set of size 1024, only 273 multiwavelets are used in the reconstruction. Multiwavelets have a high order basis approximation, which translates into smaller error between nodal points. This example demonstrates the effectiveness of the multiwavelet basis in accurately approximating functions with minimal cost – A property that we will show has a direct impact when multiwavelets are used in the DG method.

(a) (b)

Figure 2.2: (a) Error in the approximation of the function, $f(x) = \sin(\pi x)$, using all 1025 hierarchical basis. (b) Error in multiwavelet reconstruction for $k = 4$, where only 273 out of a possible 1024 basis functions are used.

3 Multiwavelet Discontinuous Galerkin for Conservation Laws

We begin introducing the DG method and how the multiwavelet scaling function can be merged into the DG method to produce an adaptive multiscale DG method. This section will provide a brief introduction to the DG method for convection dominated problems and present the notation necessary to explain a multiwavelet DG method. Further details about the DG method can be obtained from the excellent DG review article [7].

Consider the one dimensional scalar non-linear conservation law,

$$u_t + f\Big(u\Big)_x = 0. \tag{3.1}$$

When solutions of Equation 3.1 become complicated or the time of evolution become large, it is necessary to utilize high order numerical methods to obtain acceptable resolution. Advances in finite difference, finite element, and finite volume methods have produced various high order schemes for each of these numerical techniques. There are pros and cons associated with these high order methods that are well explained in [14]. This study focuses on the DG method, which is a type of finite element method that retains local conservation by incorporating the concepts of numerical fluxes and limiters developed by high order finite difference and finite volume schemes. Proving convergence of a numerical scheme for the general non-linear case of Equation 3.1 is challenging and open problem, but the DG method has the best provable stability properties in comparison to all the above numerical schemes in the sense that restricting Equation 3.1 to convex scalar conservation laws, any converged solution is an entropy solution for all orders of accuracy and all spacial dimensions, with arbitrary triangulation and without the need of nonlinear limiters [9].

The DG method is flexible in the choice of mesh and basis functions used to approximate the solution of Equation 3.1. Specifically setting the DG solution approximation space of Equation 3.1 to the space of piecewise smooth polynomials as defined in Equation 2.1, it is possible to seamlessly incorporate Multiwavelets into the DG method. Given a fixed order $k \geq 0$ and resolution $n \geq 0$ and assuming a solution of the form $u_h\Big(x,t\Big) \in \mathbf{V}_n^k$, the variational formulation of the DG method is derived by multiplying Equation 3.1 by the test functions $\phi_{jl} \in \mathbf{V}_n^k$ and integrating by parts to obtain

$$\begin{aligned}
\int_{I_{nl}} \Big(u_h\Big(x,t\Big)\Big)_t \phi_{jl}^n(x)dx &= \int_{I_l^n} f\Big(u_h\Big(x,t\Big)\Big)\Big(\phi_{jl}^n(x)\Big)_x dx \\
&- \hat{f}\Big(u_h\Big(2^n(l+1),t\Big)\Big)\phi_{jl}^n\Big(2^n(l+1)\Big) \\
&+ \hat{f}\Big(u_h\Big(2^n l,t\Big)\Big)\phi_{jl}^n\Big(2^n l\Big),
\end{aligned} \tag{3.2}$$

for $j = 0,\ldots,k-1$ and $l = 0,1,\ldots,2^n-1$, and $\hat{f}\Big(u_h\Big)$ is a monotone numerical flux. Here we adopted notation standard to multiwavelets to express the the numerical DG scheme.

Using the two scale relationship Equation 2.15 on the test functions in Equation 3.2 we can re-write the numerical DG scheme as

$$\begin{aligned}\int_0^1 \Big(u_h\Big(x,t\Big)\Big)_t \phi_j(x)dx &= \int_0^1 f\Big(u_h\Big(x,t\Big)\Big)\Big(\phi_j(x)\Big)_x dx \\ &- \hat{f}\Big(u_h\Big(x,t\Big)\Big)\phi_j^n(x)\Big|_{x=0}^{x=1},\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\int_{I_{ml}} \Big(u_h\Big(x,t\Big)\Big)_t \psi_{jl}^m(x)dx &= \int_{I_{2l}^{m+1}} f\Big(u_h\Big(x,t\Big)\Big)\Big(\psi_{jl}^m(x)\Big)_x dx \\ &+ \int_{I_{2l+1}^{m+1}} f\Big(u_h\Big(x,t\Big)\Big)\Big(\psi_{jl}^m(x)\Big)_x dx \\ &- \hat{f}\Big(u_h\Big(x,t\Big)\Big)\psi_{jl}^m(x)\Big|_{x=2^m l}^{x=2^{m-1}(l+1)} \\ &- \hat{f}\Big(u_h\Big(x,t\Big)\Big)\psi_{jl}^m(x)\Big|_{x=2^{m-1}(l+1)}^{x=2^m(l+1)},\end{aligned} \tag{3.4}$$

for $j=0,\dots,k-1$ and $l=0,1,\dots,2^m-1$ and $m=0,\dots,n-1$. Equation 3.4 is broken up into the appropriate intervals where the wavelet ψ_{jl}^m is smooth. Since multiwavelet functions are discontinuous, evaluation of the wavelet ψ_{jl}^m for the integration by parts terms are to be the limit of the wavelet as taken inside each corresponding open interval.

Transformation of the numerical DG scheme into an equivalent formulation using multiwavelets as the test functions has two immediate advantages. First, this transformation isolates a set of equations in the Equation 3.3, where the numerical flux is directly known from the boundary conditions. Second, this transformation displays, instead of the standard notion of numerical fluxes only interacting with its nearest neighbors, how the numerical flux can have a hierarchical structure that has specific interacts at various scales. Finally, we remark that the multiwavelet functions are piecewise smooth functions and the integrals are evaluated on each piecewise smooth section.

Next we assume that the approximate solution in Equations 3.3 and 3.4 have the multiwavelet expansion

$$u_h(x,t) = \sum_{j=0}^{k-1}\Big(s_{j0}^0\phi_j(x) + \sum_{m=0}^{n-1}\sum_{l=0}^{2^m-1} d_{jl}^m\psi_{jl}^m(x)\Big), \tag{3.5}$$

and as a direct result of multiwavelets orthogonality, upon substitution we have the following system of differential equations,

$$\frac{d}{dt}s_{j0}^0(t) = \int_0^1 f\Big(u_h\Big(x,t\Big)\Big)\Big(\phi_j^n(x)\Big)_x dx - \hat{f}\Big(u_h\Big(x,t\Big)\Big)\phi_j^n(x)\Big|_{x=0}^{x=1}, \tag{3.6}$$

and

$$\frac{d}{dt}d_{jl}^m(t) = \int_{I_{2l}^{m+1}} f\Big(u_h\Big(x,t\Big)\Big)\Big(\psi_{jl}^m(x)\Big)_x dx$$

$$
\begin{aligned}
&+ \int_{I_{2l+1}^{m+1}} f\Big(u_h\Big(x,t\Big)\Big)\Big(\psi_{jl}^m(x)\Big)_x dx \\
&- \hat{f}\Big(u_h\Big(x,t\Big)\Big)\psi_{jl}^m(x)\Big|_{x=2^m l}^{x=2^{m-1}(l+1)} \\
&- \hat{f}\Big(u_h\Big(x,t\Big)\Big)\psi_{jl}^m(x)\Big|_{x=2^{m-1}(l+1)}^{x=2^m(l+1)},
\end{aligned} \tag{3.7}
$$

for $j = 0, \ldots, k-1$ and $l = 0, 1, \ldots, 2^n - 1$ and $m = 0, 1, \ldots, n-1$.

The combination of multiwavelets and the discontinuous Galerkin method provides a unique combination of properties, that neither multiwavelets or discontinuous Galerkin have in isolation. Broadly speaking, using multiwavelets in discontinuous Galerkin gives the method efficient multi-scale tools. Multiwavelets have used non-standard operator forms to solve partial differential equations [4], however using the discontinuous Galerkin method with multiwavelets is a different method to solving partial differential equations that produce locally conservative and stable with high-order accuracy that can take advantage of the developments in fluxes and limiting.

4 Multiwavelet Local Discontinuous Galerkin for Convection Diffusion Problems

We demonstrate in this section how, similar to the previous section on conservation laws, incorporating multiwavelets into the local discontinuous Galerkin (LDG) method for convection diffusion problems yields an adaptive multi-scale structured and computational efficient formulation. This section will provide a brief introduction to the LDG method for convection diffusion problems and present the notation necessary to explain a multiwavelet LDG method. We refer interested readers to seminal article [6] for further information.

Consider the one dimensional convection diffusion equation,

$$
u_t + f\Big(u\Big)_x = \Big(a(u)u_x\Big)_x \qquad \text{in } (0,1) \times (0,T), \tag{4.1}
$$

and $a(u) \geq 0$. The idea of LDG is to formulate Equation 4.1 as a first order system, then use appropriate numerical fluxes that guarantee stability and local solvability of the solution approximation to the system. We begin by using the LDG approach to re-write the convection diffusion equation as the first order system,

$$
u_t + f\Big(u\Big)_x = \Big(b(u)q\Big)_x \quad \text{and} \quad q - B(u)_x = 0, \tag{4.2}
$$

where

$$
b(u) = \sqrt{(}a(u)) \quad \text{and} \quad B(u) = \int^u b(u)du. \tag{4.3}
$$

Using the same techniques as Section 3, for a fixed order $k \geq 0$ and resolution $n \geq 0$ the numerical LDG scheme is given by,

$$\begin{aligned}
\int_{I_l^n} \Big(u_h\Big(x,t\Big)\Big)_t \phi_{jl}^n(x)dx &= \int_{I_l^n} g\Big(u_h\Big(x,t\Big), q_h\Big(x,t\Big)\Big)\Big(\phi_{jl}^n(x)\Big)_x dx \\
&- \hat{g}\Big(u_h\Big(2^n(l+1),t\Big), q_h\Big(2^n(l+1),t\Big)\Big)\phi_{jl}^n\Big(2^n(l+1)\Big) \\
&+ \hat{g}\Big(u_h\Big(2^n l,t\Big), q_h\Big(2^n l,t\Big)\Big)\phi_{jl}^n\Big(2^n l\Big), \qquad (4.4)
\end{aligned}$$

and

$$\begin{aligned}
\int_{I_l^n} q_h\Big(x,t\Big)\phi_{jl}^n(x)dx &= \int_{I_l^n} B\Big(q_h\Big(x,t\Big)\Big)\Big(\phi_{jl}^n(x)\Big)_x dx \\
&- \hat{B}\Big(q_h\Big(2^n(l+1),t\Big)\Big)\phi_{jl}^n\Big(2^n(l+1)\Big) \\
&+ \hat{B}\Big(q_h\Big(2^n l,t\Big)\Big)\phi_{jl}^n\Big(2^n l\Big), \qquad (4.5)
\end{aligned}$$

for

$$g\Big(u_h\Big(x,t\Big), q_h\Big(x,t\Big)\Big) = f\Big(u_h\Big(x,t\Big)\Big) - b\Big(u_h\Big(x,t\Big)\Big)q_h\Big(x,t\Big), \qquad (4.6)$$

with $j = 0, \ldots, k-1$ and $l = 0, 1, \ldots, 2^n - 1$. Here the approximate LDG solution is $u_h, q_h \in \mathbf{V}_n^k$ with the test functions given in Equation 2.6. The numerical flux design follows the sufficient conditions described in [6] in order to guarantee the stability of the scheme.

Finally, using the two scale relationship of Equation 2.15, multiwavelets orthogonality, and assuming u_h follows the multiwavelet expansion given in Equation 3.5 and q_h is written as the multiwavelet expansion,

$$q_h(x,t) = \sum_{j=0}^{k-1} \Big(\bar{s}_{j0}^0 \phi_j(x) + \sum_{m=0}^{n-1} \sum_{l=0}^{2^m-1} \bar{d}_{jl}^m \psi_{jl}^m(x)\Big), \qquad (4.7)$$

the LDG solution to Equation 4.2 is given by the following system of differential equations,

$$\begin{aligned}
\frac{d}{dt} s_{j0}^0(t) &= \int_0^1 g\Big(u_h\Big(x,t\Big), q_h\Big(x,t\Big)\Big)\Big(\phi_j^n(x)\Big)_x dx - \hat{g}\Big(u_h\Big(1,t\Big), q_h\Big(1,t\Big)\Big)\phi_j^n(1) \\
&+ \hat{g}\Big(u_h\Big(0,t\Big), q_h\Big(0,t\Big)\Big)\phi_j^n(0), \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} d_{jl}^m(t) &= \int_{I_{2l}^{m+1}} g\Big(u_h\Big(x,t\Big), q_h\Big(x,t\Big)\Big)\Big(\psi_{jl}^m(x)\Big)_x dx \\
&+ \int_{I_{2l+1}^{m+1}} g\Big(u_h\Big(x,t\Big), q_h\Big(x,t\Big)\Big)\Big(\psi_{jl}^m(x)\Big)_x dx
\end{aligned}$$

$$
\begin{aligned}
&- \quad \hat{g}\Big(u_h\Big(x,t\Big), q_h\Big(x,t\Big)\Big)\psi_{jl}^m\Big(x\Big)\big|_{x=2^m l}^{x=2^{m-1}(l+1)} \\
&- \quad \hat{g}\Big(u_h\Big(x,t\Big), q_h\Big(x,t\Big)\Big)\psi_{jl}^m\Big(x\Big)\big|_{x=2^{m-1}(l+1)}^{x=2^m(l+1)}, \qquad (4.9)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}\bar{s}_{j0}^0(t) &= \int_0^1 B\Big(q_h\Big(x,t\Big)\Big)\Big(\phi_j^n(x)\Big)_x dx - \hat{B}\Big(q_h\Big(1,t\Big)\Big)\phi_j^n(1) \\
&+ \hat{B}\Big(q_h\Big(0,t\Big)\Big)\phi_j^n(0), \qquad (4.10)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\bar{d}_{jl}^m(t) &= \int_{I_{ml}} B\Big(q_h\Big(x,t\Big)\Big)\Big(\psi_{jl}^m(x)\Big)_x dx - \hat{B}\Big(q_h\Big(x,t\Big)\Big)\psi_{jl}^m\Big(x\Big)\big|_{x=2^m l}^{x=2^{m-1}(l+1)} \\
&+ \hat{B}\Big(q_h\Big(x,t\Big)\Big)\psi_{jl}^m\Big(x\Big)\big|_{x=2^{m-1}(l+1)}^{x=2^m(l+1)}, \qquad (4.11)
\end{aligned}
$$

for $j = 0, \ldots, k-1$ and $l = 0, 1, \ldots, 2^n - 1$ and $m = 0, 1, \ldots, n-1$. Just as in the previous section, evaluation of the wavelets for the integration by parts terms are to be the limit of the wavelets as taken inside each corresponding open interval.

5 Time Discretization

We use a method of time stepping that has been used effectively for certain multiwavelet based schemes [2, 5]. The idea behind the development of these schemes, as it is related to this research, is to convert differential equations of the form,

$$u_t = \mathcal{L}u + \mathcal{N}(u), \qquad (5.1)$$

where the system is split into a linear operator $\mathcal{L}$ and nonlinear operator $\mathcal{N}$, into the equivalent integral equation,

$$u(t) = e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}}\mathcal{N}(u)d\tau. \qquad (5.2)$$

The nature of the multiwavelet basis is used to design a fast scaling and squaring methods that produce sparse and accurate approximations to the exponential linear operator. These time-stepping schemes are therefore called *exact linear part* (ELP) schemes [2, 5].

6 Examples

To demonstrate the properties of multiwavelet DG and LDG we will begin by focusing on Examples 6.1 and 6.2, which deal with linear conservation and convection diffusion problems, respectively. Next we consider a non-linear problem in Example 6.3. We end off this section with a more challenging problem, Example 6.4 and 6.5, which are standard tests for climate models. For all examples throughout this paper we use the well known simple Lax-Friedrichs flux [10]. Additionally, we emphasize that when we mention DG we refer to the solution of Equation 3.2, where as multiwavelet DG is the solution of Equations 3.6 and 3.7. Similarly, when we mention LDG we refer to the solution of Equations 4.4 and 4.5, where as multiwavelet LDG is the solution of Equations 4.8 to 4.11.

Example 6.1 For arbitrary dimension, d, consider the linear conservation law problem

$$u_t + \nabla \cdot u = 0, \qquad \text{in } (0,1)^d \times (0,T), \tag{6.1}$$

for periodic boundary conditions and initial conditions $u_0(x) = u(0,t)$.

Example 6.2 For arbitrary dimension, d, consider the linear convection diffusion problem

$$u_t + \nabla \cdot u - \nabla \cdot \nabla u = 0, \qquad \text{in } (0,1)^d \times (0,T), \tag{6.2}$$

for periodic boundary conditions and initial conditions $u_0(x) = u(0,t)$.

	DG		Multiwavelet DG	
Resolution (n)	L_2 error	Order	L_2 error	Order
k=2				
4	1.1954e-02	-	1.1954e-02	-
5	1.1186e-03	3.42	1.1186e-03	3.42
6	1.0268e-04	3.45	1.0268e-04	3.45
7	1.0098e-05	3.35	1.0111e-05	3.35
k=3				
4	4.7328e-04	-	4.7328e-04	-
5	2.0856e-05	4.50	2.0835e-05	4.51
6	9.3186e-07	4.48	9.4271e-07	4.47
7	5.8985e-08	3.98	5.0292e-08	4.22

Table 6.1: Convergence rates for DG and multiwavelet DG for Example 6.1. Here the initial conditions are $u_0(x) = \sin(2\pi x)$, with $k = 2$ and $k = 3$ and error tolerance $\epsilon = 1e^{-8}$.

We begin by analyzing the convergence properties of the proposed numerical methods for Example 6.1 and 6.2. It can be seen in Table 6.1 and 6.2 that the expected convergence rate of $k + 1$ is demonstrated for Example 6.1 and to a lesser degree for Example 6.2. We note that in Table 6.2 the convergence rate for the convection diffusion problem falls short of the rate of $k + 1$ when the error tolerance is set at $\epsilon = 1e^{-8}$. However, this convergence rate is restored when the error tolerance is lowered. Finally, we note that the multiwavelet LDG is more robust with respect to error tolerance in maintaining its convergence rate for higher resolution and polynomial basis order.

Figure 6.1 displays the property of similar sparsity pattern for the one dimensional forward and inverse DG linear operator of Example 6.1 and LDG operator of Example 6.2.

	LDG		Multiwavelet LDG	
Resolution (n)	L_2 error	Order	L_2 error	Order
k=2				
4	1.1529e-03	-	1.1529e-03	-
5	1.4610e-04	2.98	1.4610e-04	2.98
6	2.2534e-05	2.70	2.2534e-05	2.70
7	3.8269e-06	2.56	3.8241e-06	2.56
k=3				
4	4.1736e-05	-	4.1739e-05	-
5	2.5938e-06	4.01	2.5920e-06	4.01
6	2.0724e-07	3.65	2.0067e-07	3.69
7	2.0666e-08	3.33	1.7749e-08	3.50

Table 6.2: Convergence rates for LDG and multiwavelet LDG for Example 6.2. Here the initial conditions are $u_0(x) = \sin(2\pi x)$, with $k = 2$ and $k = 3$ and error tolerance $\epsilon = 1e^{-8}$.

The forward operator represents an explicit time step and the inverse operator represents an implicit time step using the ELP scheme discussed in section 5. The inverse operators of Figure 6.1 display a generous time step, of the order of the CFL condition. It can be seen that sparsity pattern of multiwavelet DG/LDG inverse operators are self similar to their forward operators. The fact that multiwavelet DG/LDG have a predictive sparse structure for both explicit and implicit time-stepping is a highly desired property in high performance computing, allowing for the development of highly effective preconditioners for use in sparse iterative methods, producing a highly parallel computational algorithm with nearly linear scaling of memory and computational performance with increasing degrees of freedom and dimensionality.

(a) **(b)** **(c)** **(d)**

Figure 6.1: (a) Forward and (b) inverse multiwavelet DG operator for example 6.1 with $n = 8$, $k = 3$, and $d = 1$ with 41179 and 30726 non-zero elements respectively. (c) Forward and (d) inverse LDG operator for Example 6.2 with $n = 8$, $k = 3$, and $d = 1$ with 47772 and 37920 non-zero elements respectively. The time step is $\Delta t = \frac{1}{2^n k}$.

Table 6.3 shows the ratio of CPU time in the one dimensional case for DG as compared to multiwavelet DG for Example 6.1 and LDG as compared to multiwavelet DG for Example

	Resolution (n)					
	example 6.1			example 6.2		
Order (k)	7	8	9	7	8	9
1	0.86	0.98	1.07	0.86	0.98	0.96
2	0.98	1.04	1.39	0.99	1.04	1.25
3	1.19	1.36	2.31	1.33	1.52	2.36
4	2.03	3.75	4.58	2.91	4.00	5.68
5	2.95	5.61	7.36	4.28	6.07	9.14

Table 6.3: Ratio of CPU time in the one dimensional case for DG as compared to multiwavelet DG for Example 6.1 and LDG as compared to multiwavelet LDG for Example 6.2. Here the initial conditions are $u_0(x) = \sin(2\pi x)$, with error tolerance $\epsilon = 1e^{-8}$.

6.2. It can be seen that the benefits of using multiwavelets in DG increases strongly with the order and resolution of the method. The underlying phenomena behind the speed-up of the multiwavelet DG method is the property that given a specified error tolerance, the number of multiwavelet coefficients needed to represent a function, as expressed by Equation 2.13, is less than is needed for the local DG basis, as expressed by Equation 2.11. The improvement is slightly higher for the convection diffusion since the inverse operator is more dense for Example 6.2 as compared to Example 6.1, and hence the compactness of the multiwavelet representation is amplified.

		Dimension (d)					
		example 6.1			example 6.2		
Resolution (n)	Order (k)	1	2	3	1	2	3
5	1	0.33	0.32	0.36	0.37	0.37	0.38
3	2	0.65	0.82	1.12	0.67	0.88	1.17
2	3	1.04	3.96	9.22	1.08	4.81	11.28

Table 6.4: Ratio of CPU Time with increasing dimensions for DG as compared to multiwavelet DG for Example 6.1 and LDG as compared to multiwavelet LDG for example 6.2. Here the initial conditions are $u_0(\mathbf{x}) = \prod_{i=1}^{d} \sin(2\pi x_i)$, with error tolerance $\epsilon = 1e^{-8}$.

Table 6.4 extends the analysis of the ratio of CPU times to consider what occurs in higher dimensions. It is demonstrated that for both Example 6.1 and 6.2 the benefits of using multiwavelets in DG and LDG improve with the dimension of the problem and this improvement is amplified for higher orders. Again, the underlying phenomena behind the speed-up of the multiwavelet DG and LDG method is because the compactness of the multiwavelet representation relative to the local DG basis is amplified with increasing dimension, order, and resolution.

Next we consider the viscous Burgers' equation.

Example 6.3 The one dimensional Burger's equation given by,

$$u_t - \nu u_{xx} + u u_x = 0, \qquad \text{in } (0,1)^d \times (0,T), \tag{6.3}$$

(a) (b)

(c) (d)

Figure 6.2: Multiwavelet DG solution of Example 6.3 for (a) $T = 0$ and (b) $T = \frac{1}{16}$. Adaptivity pattern of the multiwavelet coefficients at (c) $T = 0$ and (d) $T = \frac{1}{16}$.

for periodic boundary conditions has the exact solution given as,

$$u = -2\nu \frac{\phi_x(x - ct, t + \tau)}{\phi(x - ct, t + \tau)} \tag{6.4}$$

for $\tau > 0$, with

$$\phi(x, t) = \sum_{n=-\infty}^{n=\infty} e^{-(x-n)^2/4\nu t}. \tag{6.5}$$

This problem displays the natural adaptively of multiwavelet DG. Here no limiting is done to preserve the shock location. Figure 6.2(b) displays the multiwavelet DG solution of Example 6.3 for $T = \frac{1}{16}$. For this solution, a third order explicit ELP scheme is used with $c = 4$, $\nu = \frac{1}{10\pi}$, $\tau = \frac{1}{2\pi}$, $k = 3$, and error tolerance $\epsilon = 1e^{-8}$. Figure 6.2(c) displays the

	LDG		Multiwavelet LDG		
Resolution (n)	L_2 error	Order	L_2 error	Order	Ratio
5	5.5290e-04	-	5.5290e-04	-	0.91
6	4.9922e-05	3.47	4.9922e-05	3.47	1.07
7	3.2172e-06	3.96	3.2168e-06	3.96	1.22

Table 6.5: Convergence rates for DG and multiwavelet DG for Example 6.3, along with ratio of CPU Time. A third order explicit ELP scheme is used with $c = 4$, $\nu = \frac{1}{10\pi}$, $\tau = \frac{1}{2\pi}$, final time $t = \frac{1}{16}$, $k = 3$, and error tolerance $\epsilon = 1e^{-8}$.

adaptivity pattern of the multiwavelet coefficients at $T = 0$, where the blue dot represents a group of k coefficients and the red line shows the support of the associated basis functions. For instances, at level zero there is one blue dot that represents the multiwavelet coefficients $\{d^0_{00}, d^0_{10}, d^0_{20}\}$ that are used in the solution approximation, where the one red line represents the support of the associated basis functions, $\{\psi^0_{00}, \psi^0_{10}, \psi^0_{20}\}$. Skipping to level two, we can see that out of the four possible groups that can be used in reconstruction only the coefficients $\{d^2_{02}, d^2_{12}, d^0_{22}\}$ and $\{d^2_{03}, d^2_{13}, d^0_{23}\}$ are used. The two lines on level two, where the starting and finishing positions are marked by red circles, depict the domain of the basis functions associated with the coefficients on this level. The number of possible groups of coefficients doubles for each increase in level. However, it can be seen in Figure 6.2(c) that the adaptivity of the multiwavelet coefficients is focused at the discontinuity of the solution. Figure 6.2(d) shows that the adaptivity changes with the movement of the solution discontinuity. Finally, Table 6.5 shows that favorable properties of multiwavelet DG demonstrated for the previous linear problems are maintained for this non-linear example.

We end this section with a couple of problems that has specific importance to the development of climate models. Specifically, convection on a sphere and the shallow water equations on the sphere. The purpose of showing these examples are two fold. First, the multiwavelet DG is capable of challenging non-linear problems. Second, the major constraint in the multiwavelet DG which is the restriction to quadrilateral elements, can be overcome and the method expanded to different geometries by developing appropriate transforms.

Example 6.4 Given the advecting field, h, the equation for convection on a sphere in flux form is,

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{v}) = 0. \tag{6.6}$$

The first test in the standardized suit developed by the climate modeling community [15] is to solve Equation 6.6 with initial conditions

$$h_0(\lambda, \theta) = \begin{cases} 1000\cos(3\pi r), & \text{if } r < \frac{1}{3}, \\ 0, & \text{else}, \end{cases} \tag{6.7}$$

for $r(\lambda, \theta) = \arccos\left(\cos(\theta)\cos\left(\lambda - \frac{3\pi}{2}\right)\right)$ and advecting wind

$$\mathbf{v} = \begin{pmatrix} \cos(\theta)\cos(\alpha) + \sin(\theta)\cos(\lambda)\sin(\alpha) \\ -\sin(\lambda)\sin(\alpha) \end{pmatrix}. \tag{6.8}$$

The design of local methods to solve problems such as Example 6.4 have received much attention since they have the potential of exploiting high-performance computing with much more efficiency than current climate models. In this paper we use the DG method developed by [11] to solve Example 6.4. We describe the major features of the DG model and refer the reader to [11] for the extensive details. The goal of this example is to demonstrate how the incorporation of multiwavelet DG into this problem has a favorable impact.

(a) **(b)** **(c)**

Figure 6.3: (a) Red points on the sphere, transfered to the blue points on the cube forms the basis in developing the so-called 'cubed-sphere' model for problems such as Example 6.4. (b) The advecting wind on the sphere of Equation 6.8 for $\alpha = \frac{\pi}{4}$, and (c) the advecting wind on the sphere transforms to the cube surface

Figure 6.3(a) depicts the geometry used to solve Example 6.4, an inscribed cube is used to develop cube-to-sphere and sphere-to-cube transformations. The convection problem is transformed onto the cube and [11] shows that the DG method used on each face of the cube with appropriate coupling is a very effective local method to solve Example 6.4. Figure 6.3(b) depicts the most computationally challenging direction for the advecting wind on the sphere (see eq. 6.8), which is when $\alpha = \frac{\pi}{4}$. Figure 6.3(c) depicts how the advecting wind on the sphere transforms to the cube surface.

Figure 6.4(a) depicts the initial conditions (6.7) on the cube surface. The direction of the advection wind takes the solution h through four corners and along two cube edges for every rotation around the cube. Figure 6.4(b) depicts the final multiwavelet DG solution after one rotation. In this simulation the order was $k = 3$, the number of elements was 364, and the error tolerance was $1e^{-8}$. In comparison to the DG solution, the multiwavelet DG ran two times faster. The major contributor to the speed-up of multiwavelet DG stems from the fact that for the specified error tolerance, the initial conditions (6.7) can be expressed with three times fewer coefficients using multiwavelets over the local DG basis.

Example 6.5 The shallow water equations on the sphere is given by,

$$\frac{\partial h^* \mathbf{v}}{\partial t} + \nabla \cdot (\mathbf{v} h \mathbf{v}) + f\hat{\mathbf{k}} \times h^* \mathbf{v} + g h^* \nabla h = 0$$

$$. \frac{\partial h^*}{\partial t} + \nabla \cdot (h^* \mathbf{v}) = 0 \qquad (6.9)$$

(a)

(b)

Figure 6.4: (a) Initial conditions (6.7) on the cube surface and the (b) Relative error after one complete revolution, with $k = 3$ and 364 elements.

where f is the Coriolis parameter, h^* is the depth of the fluid and h is the height of the free surface, or $h = h^* + h_s$ with h_s the height of the earth surface. The vector velocity is denoted as $\mathbf{v}$ with components u in the longitudinal (λ) direction and v in the latitudinal (θ) directions. Operators are given as,

$$\nabla() \equiv \frac{\mathbf{i}}{r\cos\theta}\frac{\partial}{\partial\lambda} + \frac{\mathbf{j}}{r}\frac{\partial}{\partial\theta} \tag{6.10}$$

and

$$\nabla\cdot\mathbf{v} \equiv \frac{1}{r\cos\theta}\left[\frac{\partial u}{\partial\lambda} + \frac{\partial v\cos\theta}{\partial\theta}\right], \tag{6.11}$$

with r being the radius of the earth.

The fifth test in the standardized suit developed by the climate modeling community [15] is to solve Equation 6.9 with initial conditions

$$h = 5400 + \frac{2r\Omega + 1}{2g}\Big(- \cos\lambda\cos\theta\sin\alpha + \sin\theta\cos\alpha\Big)^2, \tag{6.12}$$

and

$$h_s = 2000\Big(1 - \frac{9\min[\frac{\pi^2}{89}, (\lambda - \frac{3\pi}{2})^2 + (\theta - \frac{\pi}{6})^2]}{\pi}\Big), \tag{6.13}$$

where Ω is the rotational rate of the earth and the vector velocity is given in Equation 6.8 for $\alpha = 0$.

Figure 6.5: Shallow water equations on the sphere at day 15, with conditions given in Example 6.5 for $k = 3$ and 364 elements.

Figure 6.5 depict the shallow water equations on the sphere at day 15, with conditions given in Example 6.5 for $k = 3$ and 364 elements. We used a third order explicit ELP scheme using a time step of 22.5 mins. The results compare favorable to other published results [12].

7 Conclusions

This research demonstrates that merging the DG method with the advances in multiwavelets provides an effective adaptive multiscale DG method. We show that multiwavelet DG has greater computational efficiency and resilience to error tolerance levels. We use conservation and convection diffusion problems in multiple dimensions to highlight the partial decoupling across different length scales that occurs in multiwavelet DG. It is seen that the multiwavelet DG operator and its inverse share the same sparse pattern, an important property that we hope to exploit in future work. One current restriction of using multiwavelet DG is the method is only developed for the quad-mesh, extension to triangle-mesh should be possible using work done on the multiwavelet basis by [8]. Finally, all of the favorable properties of multiwavelet DG hold, and even are enhanced, for higher dimensional problems with large degrees of freedom, thus encouraging future work in the development of fast numerical solvers, on high performance computing, for large scale problems.

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